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# THE GENERATION OF WAVES IN AN INFINITE ELASTIC SOLID BY VARIABLE BODY FORCES

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CONTENTS



This paper is concerned with the determination of the distribution of stress in an infinite elastic solid when time-dependent body forces act upon certain regions of the solid. It is assumed throughout that the strains are small. In §2 a general solution of the equations of motion for any distribution of body forces is derived by the use of four-dimensional Fourier transforms, and from that is derived the general solution for an isotropic solid (§3). From the latter solution are deduced the general solution of the statical problem (§4) and the two-dimensional problem (§5). The solution of the equations of motion in the case in which the distribution of body forces is symmetrical about an axis is derived in §6.

The remainder of the paper consists in deducing the solution of special problems from these general solutions. In §§7 to 13 some typical two-dimensional problems are considered and exact

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analytical expressions found for the components of the stress tensor. In §§14 to 16 examples are given of the use of the general non-symmetrical three-dimensional solution derived in §3, and in §§17 to 19 examples are given to illustrate the use of the general solution of the axially symmetrical problem. A certain amount of numerical work (presented in graphical form) is quoted to give some idea of the physical nature of the solutions.

## 1. Introduction

The problem of calculating the components of the stress tensor at a point in an elastic solid, when it is deformed by the application of surface tractions which vary with the time, is of considerable importance in soil mechanics, in the theory of foundations and in other branches of applied mathematics. There has been extensive discussion of the corresponding statical problems, but it is only recently (Sneddon 1951, pp. 444–9; 1954) that attempts have been made to study dynamical problems of this type in a systematic way. Special problems have been solved by Lamb (1904), Nakano (1925), Smirnov & Sobolov (1932), Cagniard (1939), Sneddon (1952), Dix (1954) and Pinney (1954) but they do not contribute much to a general theory.

There is a similar boundary-value problem in the case in which the elastic solid is deformed by the action of external body forces. The solution of a problem of this kind when the surfaces of the elastic solid are free from stress is of great importance in the theory of seismology. Statical solutions of problems of this type have been derived by various authors (Mindlin 1936; Dean, Parsons & Sneddon 1944; Sneddon 1944, 1951, chaps. 9 and 10), and recently Lapwood (1949) has derived solutions for the deformation of a semi-infinite elastic solid by body forces which vary with the time.

The present paper is the first of a series giving detailed solutions of the problems (of both the above types) discussed by Sneddon in his Palermo lecture (Sneddon 1954). It is concerned solely with the determination of the distribution of stress in an infinite elastic solid when time-dependent body forces act on certain regions of the solid. The strains are assumed to be infinitesimal so that the equations of the classical theory of elasticity (Green & Zerna 1954, chap. v), are applicable. In §2 a general solution of the equations of motion appropriate to any distribution of body forces is derived by the use of four-dimensional Fourier transforms. From that solution is derived the general solution for an isotropic solid (§3). In §§4 and 5 the general solutions for the statical problem and for the two-dimensional problem for an isotropic solid are deduced from the solution of §3. The solution of the equations of motion in the case in which the distribution of body forces possesses axial symmetry is obtained in §6 by means of integral transforms whose kernels are of the form

$$\frac{r}{2\pi} J_\nu(\xi r)\, e^{i(\zeta z+\omega\tau)}$$

with $\nu = 0$ or 1.

The remainder of the paper consists in deducing the solution of special problems from the general solutions stated in §§3 to 6. In §§7 to 13 some typical two-dimensional problems are considered and exact analytical expressions found for the components of the stress tensor. A certain amount of computational work has been done on the basis of these solutions and the results presented graphically in order to give some idea of the nature of the solutions. It was thought to be unprofitable to give extensive tables of numerical values of the components of the stress tensor, since, in most cases, analytical expressions are given for these quantities, and it is a simple matter to calculate them if that should prove to be necessary. In many cases, too, it is difficult to see how the solutions as they stand can be applied to a practical engineering problem. A body force concentrated at a point and moving with uniform velocity through an infinite solid is not easy to envisage physically. On the other hand, the solution corresponding to such a body force may be of value in the construction of the solution of more complicated (and physically realizable) problems in much the same way as solutions corresponding to moving point charges are of value in electrodynamics. In §§ 14 to 16, examples are given of the use of the general non-symmetrical three-dimensional solution derived in § 3. In a similar way §§ 17 to 19 illustrate the general theory developed in § 6.

## I. GENERAL THEORY

### 2. The general solution of the equations of motion

We shall consider the distribution of stress in an infinite elastic medium deformed by the action of body forces which may vary with time and which are applied to certain specified regions of the medium. If we describe the position of a point in the solid by three rectangular co-ordinates $x_1$, $x_2$, $x_3$ and if $\tau^{pq}$ denotes the stress tensor, then, provided the solid is homogeneous and of density $\rho$, we may write the equations of motion in the form

$$\tau^{pq}_{,q} + \rho F^p = \rho f^p \quad (p = 1, 2, 3),$$

where $(F^1, F^2, F^3)$ denote the components of the body force at the point $(x_1, x_2, x_3)$. The acceleration of an infinitesimal element centred at this point is denoted by the vector $(f^1, f^2, f^3)$. If we introduce a displacement vector with components $(v^1, v^2, v^3)$ at such a typical point we have

$$f^p = \frac{\partial^2 v^p}{\partial t^2} \equiv c^2 \frac{\partial^2 v^p}{\partial \tau^2},$$

where $t$ denotes the time, $c$ is some characteristic velocity, and $\tau = ct$ is a space-like co-ordinate determined by the time. The equations of motion may therefore be written in the form

$$\tau^{pq}_{,q} + \rho F^p = \rho c^2 \frac{\partial^2 v^p}{\partial \tau^2}. \tag{2·1}$$

In the general case of a homogeneous solid whose elastic properties do not vary with the time the relation between the components of the stress tensor and of the displacement vector may be written in the form

$$\tau^{pq} = \tfrac{1}{2} E^{pqrs}(v_{r,s} + v_{s,r}), \tag{2·2}$$

where the quantities $E^{pqrs}$ are constants which are independent of $x_1$, $x_2$, $x_3$ and $\tau$.

The problem we shall consider here is that of solving the set of equations (2·1) and (2·2) when the mode of variation of the components $F^p$ throughout the solid is prescribed. To solve these equations we introduce the four-dimensional Fourier transform of each of the components of stress and displacement. We shall denote the Fourier transform of a function, $\phi$, by placing a bar over it, thus, $\bar{\phi}$; in other words,

$$\bar{\phi}(\xi_1, \xi_2, \xi_3, \omega) = \frac{1}{4\pi^2} \int_{E_4} \phi(x_1, x_2, x_3, \tau) \exp\{i(x_p \xi_p + \omega\tau)\} \, dV, \tag{2·3}$$

where $dV = dx_1\,dx_2\,dx_3\,d\tau$ and $E_4$ denotes the entire $x_1x_2x_3\tau$-space. If we multiply both sides of equations (2·1) and (2·2) by $\exp\{i(x_p\xi_p+\omega\tau)\}$ and integrate over $E_4$ then, making use of the results

$$\frac{1}{4\pi^2}\int_{E_4}\left(\frac{\partial\phi}{\partial x_q},\frac{\partial^2\phi}{\partial\tau^2}\right)\exp\{i(x_p\xi_p+\omega\tau)\}\,dV = -(i\xi_q,\omega^2)\,\bar{\phi}, \tag{2·4}$$

we find that the equations of motion and the stress-strain relations are equivalent to the set of algebraic equations

$$i\xi_q\bar{\tau}^{pq}-\rho\bar{F}^p = \rho c^2\omega^2\bar{v}^p, \tag{2·5}$$

$$\bar{\tau}^{pq} = -\tfrac{1}{2}iE^{pqrs}(\xi_s\bar{v}_r+\xi_r\bar{v}_s), \tag{2·6}$$

by means of which the Fourier transforms, $\bar{\tau}^{pq}$, $\bar{v}^p$, of the components of stress and displacement can be determined in terms of $\bar{F}^p$, the Fourier transforms of the components of the body force. By the symmetry properties of the constants $E^{pqrs}$ we know that $E^{pqrs}=E^{pqsr}$, so that the equations (2·6) are equivalent to the set

$$i\bar{\tau}^{pq} = E^{pqrs}\xi_r\bar{v}_s. \tag{2·7}$$

If we substitute from equations (2·7) into equations (2·5) we obtain the simple set of equations

$$\alpha^{ps}\bar{v}_s = \rho\bar{F}^p \tag{2·8}$$

for the determination of the Fourier transforms of the components of the displacement vector, where

$$\alpha^{ps} = E^{pqrs}\xi_q\xi_r-\rho c^2\omega^2\delta^{ps}, \tag{2·9}$$

in which $\delta^{ps}$ denotes the Kronecker delta.

The solution of the set of algebraic equations (2·8) is

$$\bar{v} = \rho D_s/D, \tag{2·10}$$

where

$$D_1 = \begin{vmatrix}\alpha^{12} & \alpha^{13} & \bar{F}^1\\ \alpha^{22} & \alpha^{23} & \bar{F}^2\\ \alpha^{32} & \alpha^{33} & \bar{F}^3\end{vmatrix},\quad D_2 = -\begin{vmatrix}\alpha^{11} & \alpha^{13} & \bar{F}^1\\ \alpha^{21} & \alpha^{23} & \bar{F}^2\\ \alpha^{31} & \alpha^{33} & \bar{F}^3\end{vmatrix},\quad D_3 = \begin{vmatrix}\alpha^{11} & \alpha^{12} & \bar{F}^1\\ \alpha^{21} & \alpha^{22} & \bar{F}^2\\ \alpha^{31} & \alpha^{32} & \bar{F}^3\end{vmatrix},$$

and

$$D = \begin{vmatrix}\alpha^{11} & \alpha^{12} & \alpha^{13}\\ \alpha^{21} & \alpha^{22} & \alpha^{23}\\ \alpha^{31} & \alpha^{32} & \alpha^{33}\end{vmatrix}.$$

In the above analysis we assumed that the quantities $E^{pqrs}$ were constants. Certain visco-elastic effects, observable in solids, might be taken into account by assuming that $E^{pqrs}$ were operators of the type

$$E^{pqrs} = e_0^{pqrs}+e_1^{pqrs}\frac{\partial}{\partial t}+e_2^{pqrs}\frac{\partial^2}{\partial t^2},$$

where $e_0^{pqrs}$, $e_1^{pqrs}$, ... are constants. For such a material the Fourier transforms of the components of the displacement vector are still given by the set of equations (2·10) except that now

$$\alpha^{ps} = (e_0^{pqrs}-i\omega c e_1^{pqrs}-\omega^2c^2e_2^{pqrs})\,\xi_q\xi_r-\rho c^2\omega^2\delta^{ps}. \tag{2·11}$$

This has the effect of complicating the calculations in any given problem but the principle is the same.

## 3. The solution of the equations for an isotropic solid

In the case of an isotropic elastic solid the stress-strain relation may be written in the form

$$\tau^{pq} = \lambda\Delta\delta^{pq} + \mu(v^p_{,q} + v^q_{,p}), \tag{3·1}$$

where $\Delta$ denotes the dilatation

$$\Delta = v^r_{,r}, \tag{3·2}$$

and $\lambda$ and $\mu$ are Lamé's elastic constants. For such a solid, it is readily shown that the solution of the set of equations (2·8) is

$$\bar{v}^p = \frac{\beta^2(\gamma^2-\omega^2)\,\bar{F}^p - (\beta^2-1)\,\xi^p(\xi_q \bar{F}^q)}{c_1^2(\gamma^2-\omega^2)\,(\gamma^2-\beta^2\omega^2)}, \tag{3·3}$$

where we have chosen for our characteristic velocity $c$ the velocity of propagation of $P$-waves in the solid:

$$c_1 = \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}} \tag{3·4}$$

(Bullen 1947, p. 21). In equation (3·3) we have also written

$$\gamma^2 = \xi_q\xi_q, \quad \beta^2 = \frac{\lambda+2\mu}{\mu}. \tag{3·5}$$

To obtain the corresponding expressions for the components of the displacement vector we make use of Fourier's integral theorem for four-dimensional transforms which states that if $\bar{\phi}(\xi_1, \xi_2, \xi_3, \omega)$ is defined in terms of $\phi(x_1, x_2, x_3, \tau)$ by equation (2·3) then

$$\phi(x_1, x_2, x_3, \tau) = \frac{1}{4\pi^2}\int_{W_4} \bar{\phi}(\xi_1, \xi_2, \xi_3, \omega) \exp\{-\mathrm{i}(\xi_r x_r + \omega\tau)\}\,\mathrm{d}W, \tag{3·6}$$

where $\mathrm{d}W = \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\,\mathrm{d}\omega$ and $W_4$ is the entire $\xi_1\xi_2\xi_3\omega$-space.

Inverting equation (3·3) by this rule we find that the components of the displacement vector are given by

$$v^p = \frac{1}{4\pi^2}\int_{W_4} \frac{\beta^2(\gamma^2-\omega^2)\,\bar{F}^p - (\beta^2-1)\,\xi^p(\xi_q\bar{F}^q)}{c_1^2(\gamma^2-\omega^2)\,(\gamma^2-\beta^2\omega^2)} \exp\{-\mathrm{i}(\xi_r x_r + \omega\tau)\}\,\mathrm{d}W. \tag{3·7}$$

If we denote the components of the strain tensor by $\gamma^{pq}$ then

$$\gamma_{pq} = \tfrac{1}{2}(v_{p,q} + v_{q,p}), \tag{3·8}$$

and so

$$\mathrm{i}\bar{\gamma}_{pq} = \tfrac{1}{2}(\xi_p\bar{v}_q + \xi_q\bar{v}_p). \tag{3·9}$$

Substituting from equation (3·3) into equation (3·9) we obtain the expression

$$\bar{\gamma}_{pq} = -\tfrac{1}{2}\mathrm{i}\left\{\frac{\beta^2(\gamma^2-\omega^2)\,(\xi_q\bar{F}^p + \xi_p\bar{F}^q) - 2(\beta^2-1)\,(\xi_q\xi_p)\,(\xi_r\bar{F}^r)}{c_1^2(\gamma^2-\omega^2)\,(\gamma^2-\beta^2\omega^2)}\right\} \tag{3·10}$$

for the Fourier transforms of the components of strain. Also from equation (3·2) we see that the Fourier transform of the dilatation is given by the expression

$$\bar{\Delta} = -\frac{\mathrm{i}\xi_r\bar{F}^r}{c_1^2(\gamma^2-\omega^2)}. \tag{3·11}$$



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Similarly, it follows from equation (3·1) that the Fourier transforms of the components of the stress tensor are given by the equations

$$\bar{\tau}^{pq} = \lambda\bar{\Delta}\delta^{pq} + 2\mu\bar{\gamma}^{pq}$$
$$= -\frac{\lambda \mathrm{i}\xi_r \bar{F}^r \delta^{pq}}{c_1^2(\gamma^2-\omega^2)} - \frac{\mu \mathrm{i}}{c_1^2}\left\{\frac{\beta^2(\gamma^2-\omega^2)(\xi_q\bar{F}^p+\xi_p\bar{F}^q)-2(\beta^2-1)\xi_p\xi_q(\xi_r\bar{F}^r)}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}\right\}. \quad (3{\cdot}12)$$

From this result it is readily shown that the Fourier transform of the sum of the principal stresses is given by the formula

$$\bar{\tau}^{pp} = -\frac{\mathrm{i}\rho(3\beta^2-4)\,\xi_r\bar{F}^r}{\beta^2(\gamma^2-\omega^2)}. \quad (3{\cdot}13)$$

Inverting these results by means of the Fourier integral theorem, we find that the components of the stress tensor are given by the integral formulae

$$\tau^{pq} = -\frac{\rho(\beta^2-2)}{4\pi^2\beta^2}\delta^{pq}\int_{W_4}\frac{\mathrm{i}\xi_r\bar{F}^r}{\gamma^2-\omega^2}\exp\{-\mathrm{i}(x_s\xi_s+\omega\tau)\}\,\mathrm{d}W$$
$$-\frac{\rho}{4\pi^2\beta^2}\int_{W_4}\frac{\beta^2(\gamma^2-\omega^2)(\xi_q\bar{F}^p+\xi_p\bar{F}^q)-2(\beta^2-1)\xi_p\xi_q(\xi_r\bar{F}^r)}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}\exp\{-\mathrm{i}(x_s\xi_s+\omega\tau)\}\,\mathrm{d}W, \quad (3{\cdot}14)$$

and that the sum of the principal stresses is

$$\tau^{pp} = -\frac{(3\beta^2-4)}{4\pi^2\beta^2}\rho\int_{W_4}\frac{\mathrm{i}\xi_r\bar{F}^r}{\gamma^2-\omega^2}\exp\{-\mathrm{i}(x_s\xi_s+\omega\tau)\}\,\mathrm{d}W. \quad (3{\cdot}15)$$

## 4. The solution of the statical problem

As a first application of the general theory developed in the last section, we shall obtain the solution of the statical problem. We obtain the statical problem by assuming that the body force is a function of the space variables, $x_1$, $x_2$, $x_3$, but not of the time variable $\tau$. If we write

$$F^p = \frac{1}{\rho}G^p(x_1, x_2, x_3)$$

for the components of the body force, then, making use of the result

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau = 2\pi\delta(\omega),$$

where $\delta(\omega)$ denotes the Dirac delta function of argument $\omega$, we see that the Fourier transforms of the components of the body force are

$$\bar{F}^p = \frac{\sqrt{(2\pi)}}{\rho}\bar{G}^p(\xi_1, \xi_2, \xi_3)\,\delta(\omega), \quad (4{\cdot}1)$$

where $\bar{G}^p$ is the three-dimensional Fourier transform of $G^p$. If we substitute this value for $\bar{F}^p$ into equation (3·3) we obtain the expression

$$\bar{v}^p = \sqrt{(2\pi)}\frac{\beta^2(\gamma^2-\omega^2)\bar{G}^p-(\beta^2-1)\xi^p(\xi_q\bar{G}^q)}{\rho c_1^2(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}\delta(\omega)$$
$$\equiv\sqrt{(2\pi)}\frac{\beta^2\gamma^2\bar{G}^p-(\beta^2-1)\xi^p\xi_q\bar{G}^q}{(\lambda+2\mu)\gamma^4}\delta(\omega)$$

for the Fourier transforms of the components of the displacement vector. Inverting this result by means of the Fourier inversion theorem we find that the displacement vector has components

$$v^p = \frac{1}{(2\pi)^{\frac{3}{2}}(\lambda+2\mu)}\int_{W_3}\frac{\beta^2\gamma^2 G^p-(\beta^2-1)\,\xi^p\xi_q\,\bar{G}^q}{\gamma^4}\exp\{-\mathrm{i}(x_r\xi_r)\}\,\mathrm{d}\boldsymbol{\xi},$$

where $\mathrm{d}\boldsymbol{\xi} = \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3$ and $W_3$ denotes the entire $\xi_1\xi_2\xi_3$-space.

If we now make use of the results

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{W_3}\frac{\xi_p^2\exp\{-\mathrm{i}(x_r\xi_r)\}}{(k^2+\gamma^2)^2}\,\mathrm{d}\boldsymbol{\xi} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(r^2-x_p^2-krx_p^2)\,\frac{\mathrm{e}^{-kr}}{r^3},$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{W_3}\frac{\xi_p\xi_q\exp\{-\mathrm{i}(x_r\xi_r)\}}{(k^2+\gamma^2)^2}\,\mathrm{d}\boldsymbol{\xi} = -\frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{x_p x_q}{r^3}(1+kr)\,\mathrm{e}^{-kr}, \quad (p\neq q)$$

and the Faltung theorem for three-dimensional Fourier transforms (Sneddon 1951, p. 45), we find, on letting $k\to 0$, that

$$v^p = \frac{1}{8\pi\mu\beta^2}\int_{W_3}\left\{\frac{(\beta^2+1)\,G^p(\boldsymbol{\xi})}{|\mathbf{r}-\boldsymbol{\xi}|}+\frac{(\beta^2-1)\,(x_p-\xi_p)\,(x_q-\xi_q)\,G^q(\boldsymbol{\xi})}{|\mathbf{r}-\boldsymbol{\xi}|^3}\right\}\mathrm{d}\boldsymbol{\xi} \tag{4·2}$$

is the solution of the statical problem.

In the special case in which the body force is an isolated force of magnitude $P$ acting at the origin $(0, 0, 0)$ in the direction of the $x_3$-axis we have

$$\mathbf{G} = P(0,0,1)\,\delta(x)\,\delta(y)\,\delta(z),$$

in which case (4·2) reduces to

$$\mathbf{v} = \frac{(\lambda+\mu)\,P}{8\pi\mu(\lambda+2\mu)}\left\{\frac{x_1x_3}{r^3},\ \frac{x_2x_3}{r^3},\ \frac{x_3^2}{r^3}+\frac{\lambda+3\mu}{\lambda+\mu}\frac{1}{r}\right\}, \tag{4·3}$$

in agreement with a known result (Love 1927, p. 185).

## 5. The solution of the two-dimensional problem

The solution appropriate to a two-dimensional isotropic elastic medium in which the state of stress at a point $(x_1, x_2)$ is uniquely determined by the three components $\tau^{\alpha\alpha'}(\alpha, \alpha' = 1, 2)$ of the stress tensor, and in which the body force has components $(F^1, F^2, 0)$, may be obtained from equation (3·7) by assuming that $F^\alpha$ is a function of $x_1$, $x_2$ and $\tau$ only ($\alpha = 1, 2$). We then have

$$\bar{F}^\alpha = \surd(2\pi)\,\bar{F}^\alpha(\xi_1,\xi_2,\omega)\,\delta(\xi_3), \quad (\alpha = 1, 2), \tag{5·1}$$

where $\bar{F}^\alpha$ is defined by the relation

$$\bar{F}^\alpha(\xi_1,\xi_2,\omega) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{S_3}F^\alpha(x_1,x_2,\tau)\exp\{\mathrm{i}(\xi_1x_1+\xi_2x_2+\omega\tau)\}\,\mathrm{d}\mathbf{S}, \tag{5·2}$$

in which $\mathrm{d}\mathbf{S} = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}\tau$ and $S_3$ is the entire $x_1x_2\tau$-space. Substituting from equation (5·1) into equation (3·7) with $p = \alpha = 1, 2$ we obtain the two-dimensional solution

$$v^\alpha = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{T_3}\frac{\beta^2(\xi_1^2+\xi_2^2-\omega^2)\,\bar{F}^\alpha-(\beta^2-1)\,\xi^\alpha(\xi_1\bar{F}^1+\xi_2\bar{F}^2)}{c_1^2(\xi_1^2+\xi_2^2-\omega^2)\,(\xi_1^2+\xi_2^2-\beta^2\omega^2)}\exp\{-\mathrm{i}(\xi_1x_1+\xi_2x_2+\omega\tau)\}\,\mathrm{d}\mathbf{T}, \tag{5·3}$$



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where $d\mathbf{T} = d\xi_1 d\xi_2 d\omega$ and $T_3$ is the entire $\xi_1\xi_2\omega$-space,

$$\left.\begin{aligned}\tau^{\alpha\alpha'} = &-\frac{(\beta^2-2)\,\rho}{(2\pi)^{\frac{3}{2}}\beta^2}\,\delta^{\alpha\alpha'}\int_{T_3}\frac{i(\xi_1\bar{F}^1+\xi_2\bar{F}^2)}{\xi_1^2+\xi_2^2-\omega^2}\exp\{-i(\xi_1x_1+\xi_2x_2+\omega\tau)\}\,d\mathbf{T}\\ &-\frac{\rho}{(2\pi)^{\frac{3}{2}}\beta^2}\int_{T_3}\frac{\beta^2(\xi_1^2+\xi_2^2-\omega^2)\,(\xi_\alpha\bar{F}^{\alpha'}+\xi_{\alpha'}\bar{F}^{\alpha})-2(\beta^2-1)\,\xi_\alpha\xi_{\alpha'}(\xi_1\bar{F}_1+\xi_2\bar{F}^2)}{(\xi_1^2+\xi_2^2-\omega^2)\,(\xi_1^2+\xi_2^2-\beta^2\omega^2)}\\ &\qquad\qquad\times\exp\{-i(\xi_1x_1+\xi_2x_2+\omega\tau)\}\,d\mathbf{T}\end{aligned}\right\}\quad(5\cdot4)$$

$(\alpha, \alpha' = 1, 2)$, by which to calculate the components of stress.

The shearing stresses $\tau^{\alpha 3}$ are zero and

$$\tau^{33} = \frac{\lambda}{2(\lambda+\mu)}\,(\tau^{11}+\tau^{22}). \quad (5\cdot5)$$

## 6. The solution of the equations of motion in the case of axial symmetry

In cases in which there is symmetry about a line we may choose cylindrical co-ordinates $(r, \theta, z)$ such that the $z$-axis coincides with the axis of symmetry. When there is such symmetry the displacement vector has components $(u, 0, w)$ in this system of co-ordinates. There are four non-vanishing components of stress which may be denoted, in the usual notation, by $\sigma_r$, $\sigma_\theta$, $\sigma_z$, $\tau_{rz}$ and which are related to $u$, $w$ by the equations

$$(\sigma_r, \sigma_\theta, \sigma_z) = \lambda\Delta + 2\mu\left(\frac{\partial u}{\partial r}, \frac{u}{r}, \frac{\partial w}{\partial z}\right), \quad \tau_{rz} = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right), \quad (6\cdot1)$$

where, in these co-ordinates, the dilatation is given by the expression

$$\Delta = \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}. \quad (6\cdot2)$$

When transformed to cylindrical co-ordinates, the equations of motion (2·1) reduce, in the symmetrical case, to

$$\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}+\rho F_r = (\lambda+2\mu)\,\frac{\partial^2 u}{\partial\tau^2}, \quad (6\cdot3)$$

$$\frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}+\rho F_z = (\lambda+2\mu)\,\frac{\partial^2 w}{\partial\tau^2}, \quad (6\cdot4)$$

where $(F_r, 0, F_z)$ are the components of the body force in the given co-ordinate system.

If we substitute from equations (6·1) and (6·2) into equations (6·3) and (6·4) we obtain the pair of equations

$$\beta^2\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right\}+(\beta^2-1)\,\frac{\partial^2 w}{\partial r\,\partial z}+\frac{\partial^2 u}{\partial z^2}+\frac{\rho}{\mu}F_r = \beta^2\frac{\partial^2 u}{\partial\tau^2}, \quad (6\cdot5)$$

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+(\beta^2-1)\,\frac{\partial}{\partial z}\left\{\frac{\partial u}{\partial r}+\frac{u}{r}\right\}+\beta^2\frac{\partial^2 w}{\partial z^2}+\frac{\rho}{\mu}F_z = \beta^2\frac{\partial^2 w}{\partial\tau^2}. \quad (6\cdot6)$$

If we multiply both sides of equation (6·5) by $(2\pi)^{-1}\exp\{i(\zeta z+\omega\tau)\}\,rJ_1(\xi r)$ and both sides of (6·6) by $(2\pi)^{-1}\exp\{i(\zeta z+\omega\tau)\}\,rJ_0(\xi r)$ and, in both cases, integrate over the whole $rz\tau$-space, we find that this pair of equations is equivalent to the algebraic equations

$$(\beta^2\xi^2+\zeta^2-\beta^2\omega^2)\,\bar{u}-i(\beta^2-1)\,\xi\zeta\bar{w} = \rho\bar{F}_r/\mu, \quad (6\cdot7)$$

$$i(\beta^2-1)\,\xi\zeta\bar{u}+(\xi^2+\beta^2\zeta^2-\beta^2\omega^2)\,\bar{w} = \rho\bar{F}_z/\mu, \quad (6\cdot8)$$

between the first-order Fourier–Hankel transforms $\overline{F}_r$, $\overline{u}$ defined by

$$(\overline{F}_r, \overline{u}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}z\,\mathrm{d}\tau \int_0^\infty rJ_1(\xi r)\,(F_r, u)\,\mathrm{d}r, \tag{6·9}$$

and the zero-order Fourier–Hankel transforms $\overline{F}_z$, $\overline{w}$ defined by

$$(\overline{F}_z, \overline{w}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}z\,\mathrm{d}\tau \int_0^\infty rJ_0(\xi r)\,(F_z, w)\,\mathrm{d}r. \tag{6·10}$$

Solving equations (6·7) and (6·8) for $\overline{u}$, $\overline{w}$ in terms of $\overline{F}_r$, $\overline{F}_z$ we have

$$\overline{u} = \frac{(\xi^2+\beta^2\zeta^2-\beta^2\omega^2)\,\overline{F}_r+\mathrm{i}(\beta^2-1)\,\xi\zeta\overline{F}_z}{c_1^2(\xi^2+\zeta^2-\omega^2)\,(\xi^2+\zeta^2-\beta^2\omega^2)}, \tag{6·11}$$

$$\overline{w} = \frac{(\beta^2\xi^2+\zeta^2-\beta^2\omega^2)\,\overline{F}_z-\mathrm{i}(\beta^2-1)\,\xi\zeta\overline{F}_r}{c_1^2(\xi^2+\zeta^2-\omega^2)\,(\xi^2+\zeta^2-\beta^2\omega^2)}. \tag{6·12}$$

If, for simplicity, we restrict our attention to the case in which $F_r = 0$, $F_z = Z$ and then invert equations (6·11) and (6·12) by the appropriate theorems for Fourier and Hankel transforms (Sneddon 1951, pp. 44 and 52), we find, for the components of the displacement vector,

$$u = \frac{(\beta^2-1)}{2\pi c_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{i}\exp\{-\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}\zeta\,\mathrm{d}\omega \int_0^\infty \frac{\xi^2\zeta J_1(\xi r)\,\overline{Z}\,\mathrm{d}\xi}{(\xi^2+\zeta^2-\omega^2)\,(\xi^2+\zeta^2-\beta^2\omega^2)}, \tag{6·13}$$

$$w = \frac{1}{2\pi c_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}\zeta\,\mathrm{d}\omega \int_0^\infty \frac{(\beta^2\xi^2+\zeta^2-\beta^2\omega^2)\,\xi J_0(\xi r)\,\overline{Z}\,\mathrm{d}\xi}{(\xi^2+\zeta^2-\omega^2)\,(\xi^2+\zeta^2-\beta^2\omega^2)}. \tag{6·14}$$

Integral expressions for the non-vanishing components of the stress tensor can be obtained in a similar fashion. For instance, it follows from equation (6·2) that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}z\,\mathrm{d}\tau \int_0^\infty r\Delta J_0(\xi r)\,\mathrm{d}r = -\mathrm{i}\zeta\overline{w}+\xi\overline{u},$$

so that, from equations (6·11) and (6·12), we have, as the result of inverting,

$$\Delta = -\frac{1}{2\pi c_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{i}\exp\{-\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}\zeta\,\mathrm{d}\omega \int_0^\infty \frac{\xi\zeta J_0(\xi r)\,\overline{Z}\,\mathrm{d}\xi}{\xi^2+\zeta^2-\omega^2}. \tag{6·15}$$

The sum of the three principal stresses is then given by the formula

$$\sigma_r+\sigma_\theta+\sigma_z = (3\beta^2-4)\,\mu\Delta. \tag{6·16}$$

To obtain the corresponding expression for $\sigma_z$ we multiply the third equation of the set (6·1) by $(2\pi)^{-1}\exp\{\mathrm{i}(\zeta z+\omega\tau)\}\,rJ_0(\xi r)$ and integrate to obtain

$$\sigma_z = -\frac{\mu}{2\pi c_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}\zeta\,\mathrm{d}\omega \int_0^\infty \frac{\mathrm{i}\xi\zeta J_0(\xi r)\,\{\beta^2\zeta^2+(3\beta^2-2)\,\xi^2-\beta^4\omega^2\}\,\overline{Z}\,\mathrm{d}\xi}{(\xi^2+\zeta^2-\omega^2)\,(\xi^2+\zeta^2-\beta^2\omega^2)}. \tag{6·17}$$

Similarly, the fourth equation of the set (6·1) yields the result

$$\tau_{rz} = -\frac{\mu}{2\pi c_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}\zeta\,\mathrm{d}\omega \int_0^\infty \frac{\xi^2 J_1(\xi r)\,\{\beta^2\xi^2-(\beta^2-2)\,\zeta^2-\beta^2\omega^2\}\,\overline{Z}\,\mathrm{d}\xi}{(\xi^2+\zeta^2-\omega^2)\,(\xi^2+\zeta^2-\beta^2\omega^2)}. \tag{6·18}$$





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To derive an expression for $\sigma_r$ we make use of the fact that the equation of motion (6·3) may, in the case $F_r = 0$, be written in the form

$$\frac{\partial}{\partial r}(r^2\sigma_r) = r(\sigma_r+\sigma_\theta)+(\lambda+2\mu)\,r^2\frac{\partial^2 u}{\partial \tau^2}-r^2\frac{\partial \tau_{rz}}{\partial z}. \tag{6·19}$$

Now $\sigma_r+\sigma_\theta$ can be found from equations (6·16) and (6·17) and $u$ and $\tau_{rz}$ are given by equations (6·13) and (6·18), so that we obtain, on integrating (6·19), that

$$\left.\begin{aligned}\sigma_r = &-\frac{(\beta^2-1)\,\mu}{\pi r c_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}\zeta\,\mathrm{d}\omega\int_0^\infty\frac{\mathrm{i}\zeta\xi^2 J_1(\xi r)\,\bar{Z}\,\mathrm{d}\xi}{(\xi^2+\zeta^2-\omega^2)\,(\xi^2+\zeta^2-\beta^2\omega^2)}\\ &-\frac{\mu}{2\pi c_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{i}\exp\{-\mathrm{i}(\zeta z+\omega\tau)\}\,\mathrm{d}\zeta\,\mathrm{d}\omega\int_0^\infty\frac{\xi\zeta\{(\beta^2-2)\,\zeta^2-\beta^2\xi^2-\beta^2(\beta^2-2)\,\omega^2\}J_0(\xi_r)\,\bar{Z}\,\mathrm{d}\xi}{(\xi^2+\zeta^2-\omega^2)\,(\xi^2+\zeta^2-\beta^2\omega^2)}.\end{aligned}\right\} \tag{6·20}$$

The equations (6·16) to (6·20) are sufficient for the determination of the components of the stress tensor in cases in which the body force $Z$ is prescribed.

The corresponding statical solution is easily deduced from these results by taking

$$Z = \frac{1}{\rho}f(r,z),$$

so that, by definition,

$$\bar{Z} = \frac{\sqrt{(2\pi)}}{\rho}\bar{f}(\xi,\zeta)\,\delta(\omega), \tag{6·21}$$

where

$$\bar{f}(\xi,\zeta) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\zeta z}\,\mathrm{d}z\int_0^\infty rf(r,z)\,J_0(\xi r)\,\mathrm{d}r. \tag{6·22}$$

Substituting from equation (6·21) into equations (6·13) and (6·14) we obtain the expressions

$$u = \frac{(\beta^2-1)}{(2\pi)^{\frac{1}{2}}\mu\beta^2}\int_{-\infty}^{\infty}\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\zeta z}\,\zeta\,\mathrm{d}\zeta\int_0^\infty\frac{\xi^2 J_1(\xi r)\bar{f}(\xi,\zeta)\,\mathrm{d}\xi}{(\xi^2+\zeta^2)^2}, \tag{6·23}$$

$$w = \frac{1}{(2\pi)^{\frac{1}{2}}\mu\beta^2}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\zeta z}\,\mathrm{d}\zeta\int_0^\infty\frac{\xi(\beta^2\xi^2+\zeta^2)\,J_0(\xi r)\bar{f}(\xi,\zeta)\,\mathrm{d}\xi}{(\xi^2+\zeta^2)^2}, \tag{6·24}$$

for the components of the displacement vector in the statical case, when we perform the integrations with respect to $\omega$.

If $Z$ is due to a point force of magnitude $P$ acting at the origin, then

$$f(r,z) = \frac{P}{2\pi r}\delta(r)\,\delta(z),$$

so that $\bar{f} = (2\pi)^{-\frac{3}{2}}P$ and

$$u = \frac{(\beta^2-1)\,P}{2\pi^2\mu\beta^2}\int_0^\infty\xi^2 J_1(\xi r)\,\mathrm{d}\xi\int_0^\infty\frac{\zeta\sin(\zeta z)\,\mathrm{d}\zeta}{(\xi^2+\zeta^2)^2} = \frac{(\beta^2-1)\,rzP}{8\pi\mu\beta^2(r^2+z^2)^{\frac{3}{2}}},$$

$$w = \frac{P}{2\pi^2\mu\beta^2}\int_0^\infty\xi J_0(\xi r)\,\mathrm{d}\xi\int_0^\infty\frac{(\beta^2\xi^2+\zeta^2)\cos(\zeta z)\,\mathrm{d}\zeta}{(\xi^2+\zeta^2)^2} = \frac{\{(\beta^2+1)\,r^2+2\beta^2z^2\}P}{8\pi\mu\beta^2(r^2+z^2)^{\frac{3}{2}}},$$

in agreement with the result (4·3) above. It will be noted that in this section $r$ is the cylindrical co-ordinate $(x_1^2+x_2^2)^{\frac{1}{2}}$, whereas in (4·3) it denotes $(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}$.



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## II. TWO-DIMENSIONAL PROBLEMS

### 7. Introduction

In this section we shall consider the solution of a few two-dimensional problems by means of the formulae developed in § 5 above. We shall adopt the usual notation for such problems, i.e. we shall write

$$v^1 = u, \quad v^2 = v; \quad \tau^{11} = \sigma_x, \quad \tau^{22} = \sigma_y, \quad \tau^{12} = \tau_{xy}$$

for the components of the displacement vector and the stress tensor, the co-ordinates of a typical point of the medium being denoted by $(x, y)$.

In addition, we shall assume, for the sake of simplicity, that only the $x$-component of the body force is non-zero, i.e. we shall take $F^1 = X$, $F^2 = 0$. With this assumption we find that the equations (5·3) assume the forms

$$u = \frac{1}{(2\pi)^{\frac{3}{2}} c_1^2} \int_{S_3} \frac{\{\beta^2(\xi^2+\eta^2-\omega^2)-(\beta^2-1)\,\xi^2\}}{(\xi^2+\eta^2-\omega^2)\,(\xi^2+\eta^2-\beta^2\omega^2)} \bar{X} \exp\{-\mathrm{i}(\xi x+\eta y+\omega\tau)\}\,\mathrm{d}\mathbf{S}, \tag{7·1}$$

$$v = -\frac{\beta^2-1}{(2\pi)^{\frac{3}{2}} c_1^2} \int_{S_3} \frac{\xi\eta\bar{X}\exp\{-\mathrm{i}(\xi x+\eta y+\omega\tau)\}}{(\xi^2+\eta^2-\omega^2)\,(\xi^2+\eta^2-\beta^2\omega^2)}\,\mathrm{d}\mathbf{S}, \tag{7·2}$$

where $\mathrm{d}\mathbf{S} = \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\omega$ and $S_3$ is the whole $\xi\eta\omega$-space.

Similarly, the set of equations (5·4) lead to the relations

$$\sigma_x+\sigma_y = -\frac{2(\beta^2-1)\,\rho}{(2\pi)^{\frac{3}{2}}\beta^2} \int_{S_3} \frac{\mathrm{i}\xi\bar{X}\exp\{-\mathrm{i}(\xi x+\eta y+\omega\tau)\}}{\xi^2+\eta^2-\omega^2}\,\mathrm{d}\mathbf{S}, \tag{7·3}$$

$$\sigma_x-\sigma_y = -\frac{2\rho}{(2\pi)^{\frac{3}{2}}\beta^2} \int_{S_3} \frac{\mathrm{i}\xi\{\beta^2(\xi^2+\eta^2-\omega^2)-(\beta^2-1)\,(\xi^2-\eta^2)\}}{(\xi^2+\eta^2-\omega^2)\,(\xi^2+\eta^2-\beta^2\omega^2)} \bar{X}\exp\{-\mathrm{i}(\xi x+\eta y+\omega\tau)\}\,\mathrm{d}\mathbf{S}, \tag{7·4}$$

$$\tau_{xy} = -\frac{\rho}{(2\pi)^{\frac{3}{2}}\beta^2} \int_{S_3} \frac{\mathrm{i}\eta\{\beta^2(\xi^2+\eta^2-\omega^2)-2(\beta^2-1)\,\xi^2\}}{(\xi^2+\eta^2-\omega^2)\,(\xi^2+\eta^2-\beta^2\omega^2)} \bar{X}\exp\{-\mathrm{i}(\xi x+\eta y+\omega\tau)\}\,\mathrm{d}\mathbf{S}, \tag{7·5}$$

from which the components of stress $\sigma_x$, $\sigma_y$, $\tau_{xy}$ may readily be calculated. It will be recalled that $w = \tau_{xz} = \tau_{yz} = 0$, and that, as a result of equation (5·5), $\sigma_z = (\frac{1}{2}\beta^2-1)\,(\sigma_x+\sigma_y)/(\beta^2-1)$, in the case of plane strain.

### 8. The distribution of stress produced by a periodic point force

We shall first of all consider the solution of the equations of motion when the body force $X$, acting at the origin in the direction of $x$ increasing, varies harmonically with the time with period $2\pi/p$. For such a body force we may write

$$X = \frac{F}{\rho}\,\delta(x)\,\delta(y)\,\mathrm{e}^{\mathrm{i}\lambda\tau}, \tag{8·1}$$

where
$$\lambda = \frac{p}{c_1}. \tag{8·2}$$

It follows as a result of simple integrations that

$$\bar{X} = \frac{F}{(2\pi)^{\frac{1}{2}}\rho}\,\delta(\omega+\lambda). \tag{8·3}$$

If we rewrite equations (7·1) and (7·2) in the forms

$$u = \frac{1}{(2\pi)^{\frac{3}{2}}c_1^2}\int_{S_3}\frac{\bar{X}}{\xi^2+\eta^2}\left\{\frac{\xi^2}{\xi^2+\eta^2-\omega^2}+\frac{\beta^2\eta^2}{\xi^2+\eta^2-\beta^2\omega^2}\right\}\exp\{-\mathrm{i}(\xi x+\eta y+\omega\tau)\}\,\mathrm{d}\mathbf{S}, \tag{8·4}$$

$$v = \frac{1}{(2\pi)^{\frac{3}{2}}c_1^2}\int_{S_3}\frac{\xi\eta\bar{X}}{\xi^2+\eta^2}\left\{\frac{1}{\xi^2+\eta^2-\omega^2}-\frac{\beta^2}{\xi^2+\eta^2-\beta^2\omega^2}\right\}\exp\{-\mathrm{i}(\xi x+\eta y+\omega\tau)\}\,\mathrm{d}\mathbf{S}, \tag{8·5}$$

substitute the expression (8·3) for $\bar{X}$, and perform the $\omega$-integrations, we obtain the expressions

$$u = -\frac{F\,\mathrm{e}^{\mathrm{i}\lambda\tau}}{4\pi^2\mu\beta^2}\left\{\frac{\partial^2}{\partial x^2}I(x,y,\lambda)+\beta^2\frac{\partial^2}{\partial y^2}I(x,y,\beta\lambda)\right\}, \tag{8·6}$$

$$v = -\frac{F\,\mathrm{e}^{\mathrm{i}\lambda\tau}}{4\pi^2\mu\beta^2}\left\{\frac{\partial^2}{\partial x\,\partial y}[I(x,y,\lambda)-\beta^2 I(x,y,\beta\lambda)]\right\}, \tag{8·7}$$

where we have written

$$I(x,y,\lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\{-\mathrm{i}(\xi x+\eta y)\}\,\mathrm{d}\xi\,\mathrm{d}\eta}{(\xi^2+\eta^2)\,(\xi^2+\eta^2-\lambda^2)}. \tag{8·8}$$

If we now make the change of variables defined by $\xi=\rho\cos\phi$, $\eta=\rho\sin\phi$, $x=r\cos\theta$, $y=r\sin\theta$ we find, on performing the integration with respect to $\phi$, that $I(x,y,\lambda)$ is a function of $r$ and $\lambda$ alone and that

$$\frac{\partial}{\partial r}I(x,y,\lambda) = \frac{2\pi}{\lambda^2}\int_0^\infty\left(1-\frac{\rho^2}{\rho^2-\lambda^2}\right)J_1(r\rho)\,\mathrm{d}\rho.$$

Making use of a well-known result in the theory of Bessel functions (Watson 1944, p. 424) we find that

$$\frac{\partial}{\partial r}I(x,y,\lambda) = \frac{2\pi}{\lambda^2}\left\{\frac{1}{r}-\tfrac{1}{2}\pi\mathrm{i}\lambda H_1^{(1)}(\lambda r)\right\}, \tag{8·9}$$

where $H_\nu^{(1)}(\lambda r) = J_\nu(\lambda r)+\mathrm{i}Y_\nu(\lambda r)$, $Y_\nu(\lambda r)$ denoting Weber's Bessel function of the second kind.

Substituting from the formula (8·9) into equations (8·6) and (8·7), and writing the results in terms of $t$, the time, we see that the displacement produced by a point force of this kind has components

$$u = \frac{\mathrm{i}F\,\mathrm{e}^{\mathrm{i}pt}}{4\mu\beta^2 r}\left[\frac{1}{p}\left\{c_1H_1^{(1)}\left(\frac{pr}{c_1}\right)+c_2H_1^{(1)}\left(\frac{pr}{c_2}\right)\right\}-\frac{1}{r}\left\{x^2H_2^{(1)}\left(\frac{pr}{c_1}\right)+\beta^2y^2H_2^{(1)}\left(\frac{pr}{c_2}\right)\right\}\right], \tag{8·10}$$

$$v = -\frac{\mathrm{i}F\,\mathrm{e}^{\mathrm{i}pt}\,xy}{4\mu\beta^2r^2}\left\{H_2^{(1)}\left(\frac{pr}{c_1}\right)-\beta^2H_2^{(1)}\left(\frac{pr}{c_2}\right)\right\}, \tag{8·11}$$

where $c_2$ is the second elastic wave velocity. Expressions for the components of the stress tensor may be obtained by differentiating equations (8·10) and (8·11); these will be found to be in agreement with those found by Lamb (1904).

## 9. The distribution of stress produced by an impulsive point force

We may represent an impulsive force of magnitude $F$ acting at the origin by the expression

$$X = \frac{F}{\rho}\,\delta(x)\,\delta(y)\,\delta(t), \tag{9·1}$$

which gives us, for $\bar{X}$, the relation

$$\bar{X} = \frac{Fc_1}{\rho(2\pi)^{\frac{3}{2}}}, \tag{9·2}$$

since $\delta(t) = c_1\delta(\tau)$.

If we substitute the expression (9·2) into equations (8·4) and (8·5) we find that the components of the displacement vector, due to an impulsive force at the origin, are given by

$$u = -\frac{Fc_1}{8\pi^3\mu\beta^2}\left\{\frac{\partial^2 I_1}{\partial x^2} + \beta^2\frac{\partial^2 I_2}{\partial y^2}\right\}, \quad v = -\frac{Fc_1}{8\pi^3\mu\beta^2}\frac{\partial^2}{\partial x\,\partial y}(I_1 - \beta^2 I_2), \tag{9·3}$$

where

$$I_1 = \int_{S_3}\frac{\exp\{-\mathrm{i}(\xi x + \eta y + \omega\tau)\}\,\mathrm{d}\mathbf{S}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\omega^2)}, \quad I_2 = \int_{S_3}\frac{\exp\{-\mathrm{i}(\xi x + \eta y + \omega\tau)\}\,\mathrm{d}\mathbf{S}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\beta^2\omega^2)}. \tag{9·4}$$

As in the previous section we find that $\partial I_2/\partial r$ is a function of $r$ only, and that

$$\begin{aligned}\frac{\partial I_2}{\partial r} &= -\frac{4\pi^2}{\beta}\int_0^\infty \frac{\sin(\rho\tau/\beta)}{\rho}J_1(\rho r)\,\mathrm{d}\rho \\ &= \begin{cases} -\dfrac{4\pi^2\tau}{\beta^2 r} & (\tau \leqslant \beta r), \\ -\dfrac{4\pi^2}{\beta^2 r}\{\tau - \sqrt{(\tau^2-\beta^2 r^2)}\} & (\tau \geqslant \beta r). \end{cases}\end{aligned} \tag{9·5}$$

A similar expression can be obtained for $\partial I_1/\partial r$ by putting $\beta = 1$ in equation (9·5). Substituting these values into equations (9·3) we obtain the formulae

$$\frac{2\pi\mu\beta^2 u}{c_1 F} = \begin{cases} 0 & (r > \tau) \\ \dfrac{x^2}{r^2}(\tau^2-r^2)^{-\frac{1}{2}} + \dfrac{x^2-y^2}{r^4}(\tau^2-r^2)^{\frac{1}{2}} & (\tau' < r < \tau), \\ \dfrac{x^2}{r^2}(\tau^2-r^2)^{-\frac{1}{2}} + \dfrac{\beta y^2}{r^2}(\tau'^2-r^2)^{-\frac{1}{2}} + \dfrac{x^2-y^2}{r^4}\{(\tau^2-r^2)^{\frac{1}{2}} - \beta(\tau'^2-r^2)^{\frac{1}{2}}\} & (r < \tau'), \end{cases} \tag{9·6}$$

and

$$\frac{2\pi\mu\beta^2 v}{c_1 F} = \begin{cases} 0 & (r > \tau), \\ \dfrac{xy}{r^2}\left\{(\tau^2-r^2)^{-\frac{1}{2}} + \dfrac{2}{r^2}(\tau^2-r^2)^{\frac{1}{2}}\right\} & (\tau' < r < \tau), \\ \dfrac{xy}{r^2}\{(\tau^2-r^2)^{-\frac{1}{2}} - \beta(\tau'^2-r^2)^{-\frac{1}{2}} + \dfrac{2}{r^2}[(\tau^2-r^2)^{\frac{1}{2}} - \beta(\tau'^2-r^2)^{\frac{1}{2}}]\} & (r < \tau), \end{cases} \tag{9·7}$$

for the determination of the components of the displacement vector. In these formulae $\tau = c_1 t$ and $\tau' = c_2 t = c_1 t/\beta$.

Substituting the expressions for $u$ and $v$ into the stress-strain relations we obtain the following equations by means of which the components of stress can be calculated:

$$\frac{2\pi\beta^2(\sigma_x+\sigma_y)}{(\beta^2-1)\,c_1 F} = \begin{cases} 0 & (r > \tau), \\ x(\tau^2-r^2)^{-\frac{3}{2}} & (r < \tau); \end{cases} \tag{9·8}$$

$$\frac{\pi\beta^2(\sigma_x-\sigma_y)}{xc_1F} = \begin{cases} 0 & (r>\tau), \\ \frac{x^2-y^2}{r^2}(\tau^2-r^2)^{-\frac{3}{2}}-\frac{2(x^2-3y^2)}{r^4}(\tau^2-r^2)^{-\frac{1}{2}}-\frac{4}{r^6}(x^2-3y^2)(\tau^2-r^2)^{\frac{1}{2}} & (\tau'<r<\tau), \\ (\tau^2-r^2)^{-\frac{3}{2}}-\frac{2y^2}{r^2}\{(\tau^2-r^2)^{-\frac{3}{2}}-\beta(\tau'^2-r^2)^{-\frac{3}{2}}\}-\frac{2}{r^4}(x^2-3y^2) & \\ \quad\times\{(\tau^2-r^2)^{-\frac{1}{2}}-\beta(\tau'^2-r^2)^{-\frac{1}{2}}\}-\frac{4}{r^6}(x^2-3y^2)\{(\tau^2-r^2)^{\frac{1}{2}}-\beta(\tau'^2-r^2)^{\frac{1}{2}}\} & (r<\tau'); \end{cases} \tag{9·9}$$

$$\frac{2\pi\beta^2\tau_{xy}}{yc_1F} = \begin{cases} 0 & (r>\tau), \\ \frac{2x^2}{r^2}(\tau^2-r^2)^{-\frac{3}{2}}+\frac{2(y^2-3x^2)}{r^4}(\tau^2-r^2)^{-\frac{1}{2}}+\frac{4}{r^6}(y^2-3x^2)(\tau^2-r^2)^{\frac{1}{2}} & (\tau'<r<\tau), \\ \beta(\tau'^2-r^2)^{-\frac{3}{2}}+\frac{2x^2}{r^2}\{(\tau^2-r^2)^{-\frac{3}{2}}-\beta(\tau'^2-r^2)^{-\frac{3}{2}}\}+\frac{2(y^2-3x^2)}{r^4} & \\ \quad\times\{(\tau^2-r^2)^{-\frac{1}{2}}-\beta(\tau'^2-r^2)^{-\frac{1}{2}}\}-\frac{4}{r^6}(x^2-3y^2)\{(\tau^2-r^2)^{\frac{1}{2}}-\beta(\tau'^2-r^2)^{\frac{1}{2}}\} & (r<\tau'). \end{cases} \tag{9·10}$$

It is readily seen from these expressions that the disturbance is propagated outwards from the centre with velocities $c_1$ and $c_2 = c_1/\beta$. These waves are known in seismology as the *P*- and *S*-waves respectively (Bullen 1947, p. 74). The wave fronts are circles, centre the origin and radii $\tau = c_1 t$, $\tau' = c_2 t$. At the wave front the components of stress and of displacement have infinite discontinuities. This is, of course, an impossible situation to arise in a perfectly elastic solid; it is due to the representation of the impulsive applied force by the idealized Dirac delta function.

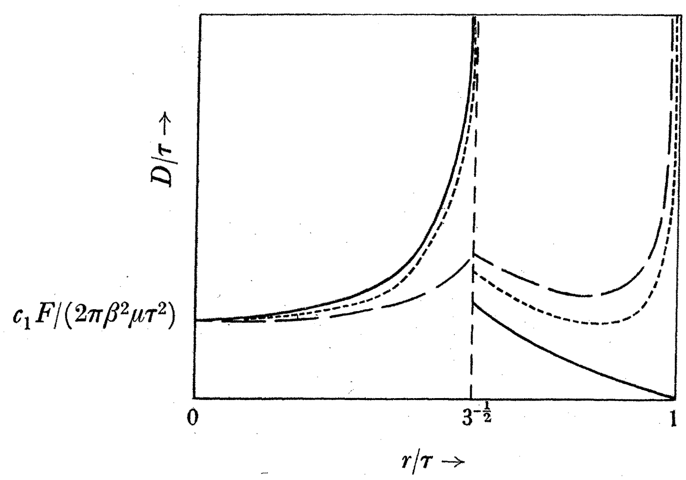


FIGURE 1. The displacement $D$ in the $r$-direction at the point with polar co-ordinates $r$, $\theta$, produced by an impulsive force $F$ applied at the origin at time $t=0$ in an infinite two-dimensional elastic solid of Poisson ratio $\frac{1}{4}$ (i.e. $\lambda = \mu$). The full curve corresponds to $\theta = 0°$, the dotted curve to $\theta = 45°$, and the broken curve to $\theta = 90°$.

An interesting fact emerges from the evaluation of the expressions for the displacement components. The variation of these components is shown in figure 1. In the direction in which the force is acting the wave front of the $P$-wave is an infinite discontinuity, but not so the wave front of the $S$-wave, while in a direction perpendicular to this, the wave front of the $S$-wave is an infinite discontinuity, but not the wave front of the $P$-wave. In directions intermediate to these, both wave fronts are infinite discontinuities. This fact may explain some of the discrepancies existing in the interpretation of geophysical observations, since in the first case, the arrival of the $S$-wave, and in the second that of the $P$-wave would not be apparent.

## 10. The effect of a point force suddenly applied

Another example which may be treated in the same way, and which can be reduced to a problem whose solution is known, thus providing a check on the method, is that of calculating the stress distribution due to an applied force of magnitude $F$, acting at the origin, whose time variation can be represented by the Heaviside unit function

$$H(t) = \begin{cases} 0 & (t<0), \\ 1 & (t>0). \end{cases}$$

We can write for the applied force

$$X = \frac{F}{\rho}\,\delta(x)\,\delta(y)\,H(t),$$

and, by using a method similar to that employed by Sneddon (1951, p. 406), we have

$$\bar{X} = \frac{F}{\rho\sqrt{(2\pi)}}\left\{\tfrac{1}{2}\delta(\omega) - \frac{1}{2\pi\mathrm{i}\omega}\right\}. \tag{10·1}$$

Substituting from equation (10·1) into equations (8·4) and (8·5) we find that the components of the displacement vector are given by the equations (9·3) with

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\{-\mathrm{i}(\xi x+\eta y)\}\,\mathrm{d}\xi\,\mathrm{d}\eta}{(\xi^2+\eta^2)^2} + \frac{\mathrm{i}}{\pi}\int_{S_3} \frac{\exp\{-\mathrm{i}(\xi x+\eta y+\omega\tau)\}\,\mathrm{d}\mathbf{S}}{\omega(\xi^2+\eta^2)\,(\xi^2+\eta^2-\omega^2)}, \tag{10·2}$$

$$I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\{-\mathrm{i}(\xi x+\eta y)\}\,\mathrm{d}\xi\,\mathrm{d}\eta}{(\xi^2+\eta^2)^2} + \frac{\mathrm{i}}{\pi}\int_{S_3} \frac{\exp\{-\mathrm{i}(\xi x+\eta y+\omega\tau)\}\,\mathrm{d}\mathbf{S}}{\omega(\xi^2+\eta^2)\,(\xi^2+\eta^2-\beta^2\omega^2)}. \tag{10·3}$$

The derivatives of these integrals $I_1$, $I_2$ may be evaluated by methods similar to those employed above to give the expressions

$$\frac{4\pi\mu\beta^2 r^2 u}{F} = \begin{cases} 0 & (r>\tau), \\ \left(1-\dfrac{2y^2}{r^2}\right)\tau(\tau^2-r^2)^{\frac{1}{2}} + r^2\ln\left\{\dfrac{\tau}{r}+\left(\dfrac{\tau^2}{r^2}-1\right)^{\frac{1}{2}}\right\} & (\tau'<r<\tau), \\ \left(1-\dfrac{2y^2}{r^2}\right)\tau\{(\tau^2-r^2)^{\frac{1}{2}}-\beta(\tau'^2-r^2)^{\frac{1}{2}}\} & \\ \quad + r^2\left[\ln\left\{\dfrac{\tau}{r}+\left(\dfrac{\tau^2}{r^2}-1\right)^{\frac{1}{2}}\right\}+\beta^2\ln\left\{\dfrac{\tau'}{r}+\left(\dfrac{\tau'^2}{r^2}-1\right)^{\frac{1}{2}}\right\}\right] & (r<\tau'); \end{cases} \tag{10·4}$$

$$\frac{2\pi\mu\beta^2 r^4 v}{F\tau xy} = \begin{cases} 0 & (r>\tau), \\ (\tau^2-r^2)^{\frac{1}{2}} & (\tau'<r<\tau), \\ (\tau^2-r^2)^{\frac{1}{2}}-\beta(\tau'^2-r^2)^{\frac{1}{2}} & (r<\tau'). \end{cases} \tag{10·5}$$

for the components of the displacement vector.

Substituting these expressions into the stress-strain relations we obtain the equations

$$\sigma_x+\sigma_y=\begin{cases}0 & (r>\tau),\\ -\dfrac{(\beta^2-1)\,\tau xF}{\pi\beta^2 r^2}(\tau^2-r^2)^{-\frac{1}{2}} & (r<\tau);\end{cases} \tag{10·6}$$

$$\frac{\pi\beta^2 r^4(\sigma_y-\sigma_x)}{x\tau F}=\begin{cases}0 & (r>\tau),\\ \left(1-\dfrac{2y^2}{r^2}\right)(\tau^2-r^2)^{-\frac{1}{2}}+\dfrac{2}{r^2}(\tau^2-r^2)^{\frac{1}{2}} & (\tau'<r<\tau),\\ (\tau^2-r^2)^{-\frac{1}{2}}-\dfrac{2y^2}{r^2}\{(\tau^2-r^2)^{-\frac{1}{2}}-\beta(\tau'^2-r^2)^{-\frac{1}{2}}\} & \\ \quad+\dfrac{2}{r^2}\left(1-\dfrac{4y^2}{r^2}\right)\{(\tau^2-r^2)^{\frac{1}{2}}-\beta(\tau'^2-r^2)^{\frac{1}{2}}\} & (r<\tau');\end{cases} \tag{10·7}$$

$$-\frac{2\pi\beta^2 r^4\tau_{xy}}{y\tau F}=\begin{cases}0 & (r>\tau),\\ \dfrac{2x^2}{r^2}(\tau^2-r^2)^{-\frac{1}{2}}-\dfrac{2}{r^2}\left(1-\dfrac{4x^2}{r^2}\right)(\tau^2-r^2)^{\frac{1}{2}} & (\tau'<r<\tau),\\ \beta(\tau'^2-r^2)^{-\frac{1}{2}}-\dfrac{2x^2}{r^2}\{\beta(\tau'^2-r^2)^{-\frac{1}{2}}-(\tau^2-r^2)^{-\frac{1}{2}}\}+\dfrac{2}{r^2}\left(1-\dfrac{4x^2}{r^2}\right) & \\ \quad\times\{\beta(\tau'^2-r^2)^{\frac{1}{2}}-(\tau^2-r^2)^{\frac{1}{2}}\} & (r<\tau').\end{cases} \tag{10·8}$$

If we let $\tau$ tend to infinity in these expressions we find that

$$\sigma_x+\sigma_y=-\frac{(\beta^2-1)\,xF}{\pi\beta^2 r^2},\quad \sigma_y-\sigma_x=\frac{xF}{\pi r^2}\left(1-\frac{\beta^2-1}{\beta^2}\frac{2y^2}{r^2}\right),\quad \tau_{xy}=-\frac{yF}{2\pi r^2\beta^2}\left\{1+\frac{2(\beta^2-1)\,x^2}{r^2}\right\}, \tag{10·9}$$

in agreement with the expressions obtained from equilibrium theory.

## 11. The stresses produced by a point force moving with uniform velocity along the line in which it acts

We shall consider the case in which the point force moves along the $x$-axis with uniform velocity $v$ and has magnitude $F$. In this case the body force may be represented by

$$X=\frac{F}{\rho}\delta(x-\alpha_1\tau)\,\delta(y), \tag{11·1}$$

where

$$\alpha_1=v/c_1. \tag{11·2}$$

We now have for $\bar{X}$ the expression

$$\bar{X}=\frac{F}{\rho\sqrt{(2\pi)}}\delta(\omega+\alpha_1\xi). \tag{11·3}$$

Equations (7·3), (7·4) and (7·5) now take the forms

$$\sigma_x+\sigma_y=-\frac{(\beta^2-1)\,F}{2\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{i}\xi\exp[-\mathrm{i}\{\xi(x-vt)+\eta y\}]\,\mathrm{d}\xi\,\mathrm{d}\eta}{\eta^2+(1-\alpha_1^2)\,\xi^2}, \tag{11·4}$$

$$\sigma_x-\sigma_y=-\frac{F}{2\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{i}\xi[\beta^2\{\eta^2+(1-\alpha_1^2)\,\xi^2\}-(\beta^2-1)(\xi^2-\eta^2)]}{\{\eta^2+(1-\alpha_1^2)\,\xi^2\}\{\eta^2+(1-\alpha_2^2)\,\xi^2\}}\exp[-\mathrm{i}\{\xi(x-vt)+\eta y\}]\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{11·5}$$

$$\tau_{xy}=-\frac{F}{4\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{i}\eta[\beta^2\{\eta^2+(1-\alpha_1^2)\,\xi^2\}-2(\beta^2-1)\,\xi^2]}{\{\eta^2+(1-\alpha_1^2)\,\xi^2\}\{\eta^2+(1-\alpha_2^2)\,\xi^2\}}\exp[-\mathrm{i}\{\xi(x-vt)+\eta y\}]\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{11·6}$$

where

$$\alpha_2=v/c_2. \tag{11·7}$$

Performing the integration with respect to $\eta$ in equation (11·4) we find that

$$\sigma_x+\sigma_y = -\frac{(\beta^2-1)F}{2\pi\beta^2(1-\alpha_1^2)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{\mathrm{i}\xi\exp\{-\mathrm{i}\xi(x-vt)-|\xi|(1-\alpha_1^2)^{\frac{1}{2}}y\}}{|\xi|}\mathrm{d}\xi$$

$$= -\frac{(\beta^2-1)F}{\pi\beta^2(1-\alpha_1^2)^{\frac{1}{2}}}\int_0^{\infty}\exp\{-(1-\alpha_1^2)^{\frac{1}{2}}\xi y\}\sin\{\xi(x-vt)\}\,\mathrm{d}\xi.$$

The evaluation of this integral is elementary and gives

$$\sigma_x+\sigma_y = -\frac{(\beta^2-1)(x-vt)F}{\pi\beta^2(1-\alpha_1^2)^{\frac{1}{2}}\{(x-vt)^2+(1-\alpha_1^2)y^2\}}. \tag{11·8}$$

Similarly the equations (11·5) and (11·6) lead to the relations

$$\sigma_x-\sigma_y = -\frac{2(x-vt)F}{\pi\alpha_2^2}\left\{\frac{(1-\frac{1}{2}\alpha_1^2)(1-\alpha_1^2)^{-\frac{1}{2}}}{(x-vt)^2+(1-\alpha_1^2)y^2}-\frac{(1-\alpha_2^2)^{\frac{1}{2}}}{(x-vt)^2+(1-\alpha_2^2)y^2}\right\}, \tag{11·9}$$

$$\tau_{xy} = -\frac{Fy}{\pi\alpha_2^2}\left\{\frac{(1-\alpha_1^2)^{\frac{1}{2}}}{(x-vt)^2+(1-\alpha_1^2)y^2}-\frac{(1-\frac{1}{2}\alpha_2^2)(1-\alpha_2^2)^{\frac{1}{2}}}{(x-vt)^2+(1-\alpha_2^2)y^2}\right\}. \tag{11·10}$$

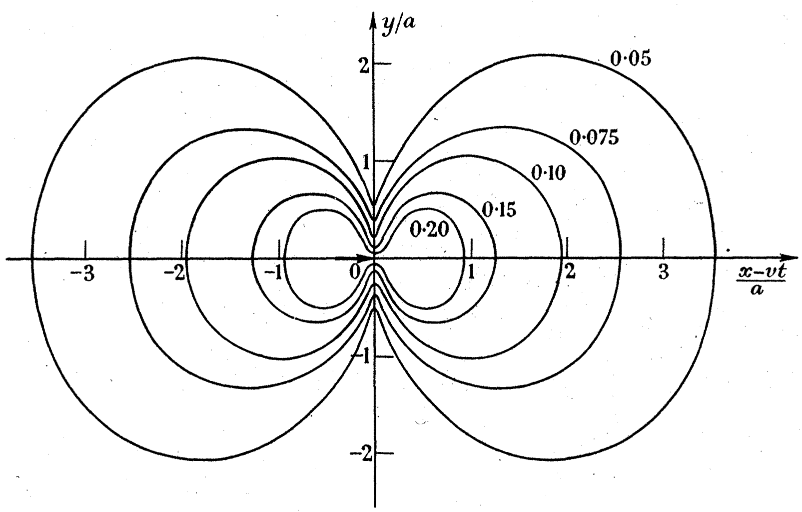


FIGURE 2. The variation of the maximum shearing stress $\boldsymbol{\tau}$ in the case of a point force of magnitude $F$ moving with uniform velocity $v = 0{\cdot}4c_1$ in the direction in which it acts in a medium with Poisson ratio $\frac{1}{4}$. The numbers attached to the curves give the values of the quantity $\pi a\boldsymbol{\tau}/F$, where $a$ is some characteristic unit of length.

These expressions are in a form suitable for calculation. The results of such a calculation are shown in figure 2, which gives a graphical representation of the maximum shearing stress $\boldsymbol{\tau}$ in the case of a point force of magnitude $F$ moving with velocity $v = 0{\cdot}4c_1$ in a medium with Poisson ratio $\frac{1}{4}$ (i.e. $\lambda=\mu$). The curves shown are the isochromatic lines $\boldsymbol{\tau}=constant$; the numbers attached to the curve give the values of the quantity $\pi a\boldsymbol{\tau}/F$, where $a$ is some characteristic length.

The corresponding curves for the statical case $v = 0$ may be obtained by letting $v\to 0$ in equations (11·8) to (11·10). They are shown in figure 3. A comparison between the two sets of curves shows clearly the effect upon the stress distribution of the velocity with which the point force moves.

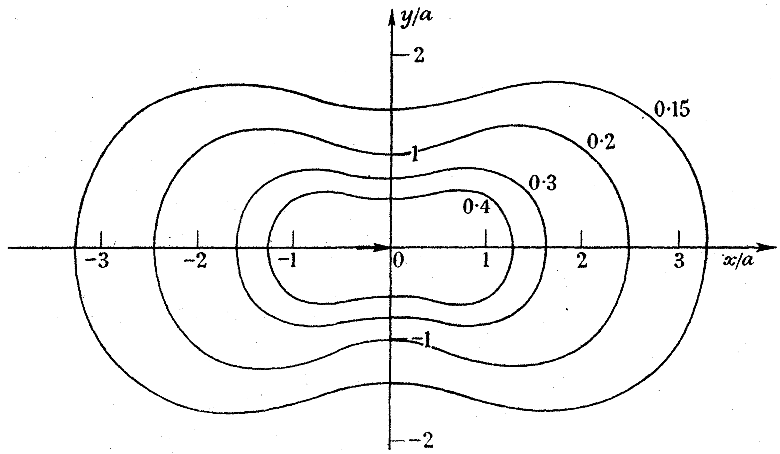


FIGURE 3. The variation of the maximum shearing stress $\tau$ in the case of a static point force of magnitude $F$ acting at the origin in a medium with Poisson ratio $\frac{1}{4}$. The numbers attached to the curves give the values of the quantity $\pi a\tau/F$, where $a$ is some characteristic length.

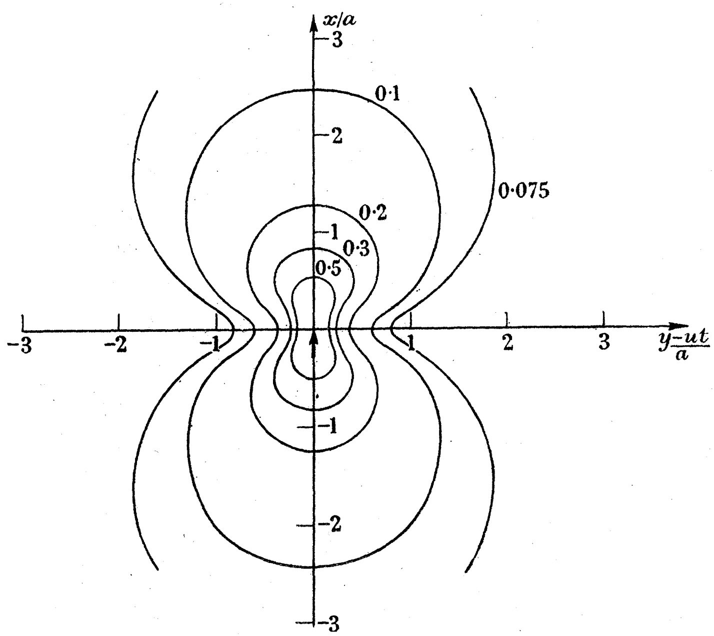


FIGURE 4. The variation of the maximum shearing stress $\tau$ in the case of a point force of magnitude $P$ moving with uniform velocity $u = 0{\cdot}4c_1$ at right angles to the direction of the force in a medium with Poisson ratio $\frac{1}{4}$. The numbers attached to the curves give the values of the quantity $\pi a\tau/P$, where $a$ is some characteristic length.

## 12. The stresses produced by a point force moving with uniform velocity at right angles to the direction in which it acts

In this section we consider a problem similar to that discussed in the last section, the difference being that now the force, which will be taken to be of magnitude $P$ pointing in the $x$-direction, moves along the $y$-axis with uniform velocity $u$. We therefore have

$$X = \frac{P}{\rho}\,\delta(x)\,\delta(y-\beta_1\tau), \tag{12·1}$$


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with $$\beta_1 = u/c_1, \tag{12·2}$$

so that $$\bar{X} = \frac{P}{\rho\sqrt{(2\pi)}}\,\delta(\omega+\beta_1\eta). \tag{12·3}$$

If we substitute for $\bar{X}$ from equation (12·3) into equation (7·3) and perform the integration with respect to $\omega$ we find that

$$\sigma_x+\sigma_y = -\frac{(\beta^2-1)\,P}{2\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{i}\xi\exp\left[-\mathrm{i}\{\xi x+\eta(y-ut)\}\right]\mathrm{d}\xi\,\mathrm{d}\eta}{\xi^2+(1-\beta_1^2)\,\eta^2}.$$

The integrations are practically identical with those of the last section. They yield the equation

$$\sigma_x+\sigma_y = -\frac{(\beta^2-1)\,(1-\beta_1^2)^{\frac{1}{2}}\,xP}{\pi\beta^2\{(1-\beta_1^2)\,x^2+(y-ut)^2\}}. \tag{12·4}$$

Similarly, from equations (7·4) and (7·5) we obtain the equations

$$\sigma_x-\sigma_y = -\frac{2Px}{\pi\beta_2^2}\left\{\frac{(1-\beta_2^2)^{\frac{1}{2}}}{(1-\beta_2^2)\,x^2+(y-ut)^2}-\frac{(1-\beta_1^2)^{\frac{1}{2}}\,(1-\frac{1}{2}\beta_1^2)}{(1-\beta_1^2)\,x^2+(y-ut)^2}\right\}, \tag{12·5}$$

$$\tau_{xy} = -\frac{(y-ut)\,P}{\pi\beta_2^2}\left\{\frac{(1-\frac{1}{2}\beta_2^2)\,(1-\beta_2^2)^{-\frac{1}{2}}}{(1-\beta_2^2)\,x^2+(y-ut)^2}-\frac{(1-\beta_1^2)^{\frac{1}{2}}}{(1-\beta_1^2)\,x^2+(y-ut)^2}\right\}, \tag{12·6}$$

with $\beta_2 = u/c_2$.

The isochromatics corresponding to a velocity $u = 0{\cdot}4c_1$ have been calculated in this case also. They are shown in figure 4.

## 13. The stresses produced by a point force moving with uniform velocity greater than $c_2$

In this section we shall calculate the stress distribution set up by a point force whose point of application is moving with uniform velocity $v > c_2$, the smaller of the wave velocities, along the line in which the force acts. As in § 11 we have $X = (F/\rho)\,\delta(x-\alpha_1\tau)\,\delta(y)$, where $\alpha_1 = v/c_1 < 1$. This results in the same expression for $X$ and equations (11·4), (11·5) and (11·6) become now

$$\sigma_x+\sigma_y = -\frac{(\beta^2-1)\,F}{2\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{i}\xi\exp\left[-\mathrm{i}\{\xi(x-vt)+\eta y\}\right]}{\eta^2+(1-\alpha_1^2)\,\xi^2}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{13·1}$$

$$\sigma_x-\sigma_y = -\frac{F}{2\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{i}\xi\{\beta^2[(1-\alpha_1^2)\,\xi^2+\eta^2]-(\beta^2-1)\,(\xi^2-\eta^2)\}\exp\left[-\mathrm{i}\{\xi(x-vt)+\eta y\}\right]}{\{\eta^2+(1-\alpha_1^2)\,\xi^2\}\{\eta^2-(\alpha_2^2-1)\,\xi^2\}}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{13·2}$$

$$\tau_{xy} = -\frac{F}{4\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{i}\eta\{\beta^2[(1-\alpha_1^2)\,\xi^2+\eta^2]-2(\beta^2-1)\,\xi^2\}\exp\left[-\mathrm{i}\{\xi(x-vt)+\eta y\}\right]}{\{\eta^2+(1-\alpha_1^2)\,\xi^2\}\{\eta^2-(\alpha_2^2-1)\,\xi^2\}}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{13·3}$$

it being assumed that $\alpha_1 < 1 < \alpha_2$.

These integrals may be evaluated by the methods outlined in § 11; we find that

$$\sigma_x+\sigma_y = -\frac{(\beta^2-1)\,(x-vt)\,F}{\pi\beta^2(1-\alpha_1^2)^{\frac{1}{2}}\{(x-vt)^2+(1-\alpha_1^2)\,y^2\}}, \tag{13·4}$$

$$\sigma_x-\sigma_y = -\frac{2F}{\pi\alpha_2^2}\left\{\frac{(1-\frac{1}{2}\alpha_1^2)\,(1-\alpha_1^2)^{-\frac{1}{2}}(x-vt)}{(x-vt)^2+(1-\alpha_1^2)\,y^2}-\tfrac{1}{2}\pi\delta\{x-vt-(\alpha_2^2-1)^{\frac{1}{2}}y\}+\tfrac{1}{2}\pi\delta\{x-vt+(\alpha_2^2-1)^{\frac{1}{2}}y\}\right\}, \tag{13·5}$$

$$\tau_{xy} = -\frac{F}{\pi\alpha_2^2}\left\{\frac{(1-\alpha_1^2)^{\frac{1}{2}}y}{(x-vt)^2+(1-\alpha_1^2)\,y^2}-\tfrac{1}{2}\pi(1-\tfrac{1}{2}\alpha_2^2)\,\delta\{(\alpha_2^2-1)^{\frac{1}{2}}y-(x-vt)\}\right.$$
$$\left.-\tfrac{1}{2}\pi(1-\tfrac{1}{2}\alpha_2^2)\,\delta\{(\alpha_2^2-1)^{\frac{1}{2}}y+(x-vt)\}\right\}. \tag{13·6}$$

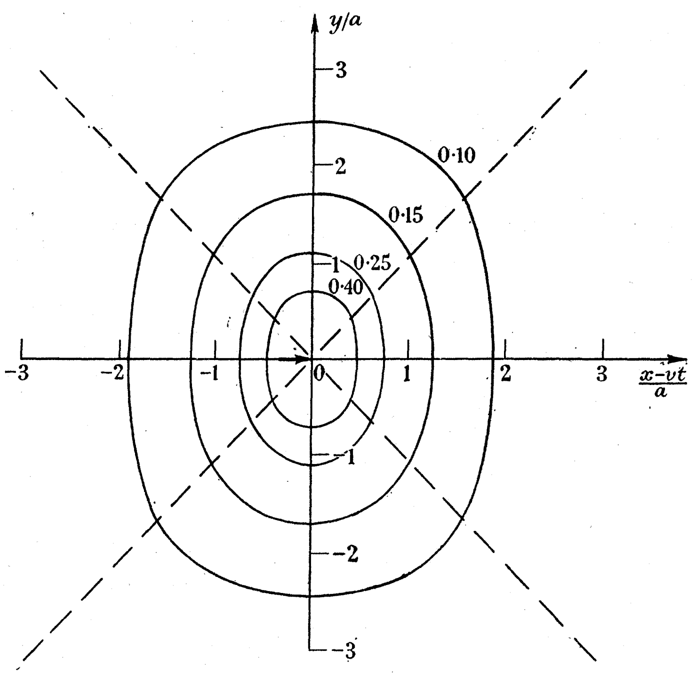


FIGURE 5. The variation of the maximum shearing stress $\tau$ in the case of a point force of magnitude $F$ moving with velocity $v = 0{\cdot}8c_1$ in the direction in which it acts in a medium with Poisson ratio $\frac{1}{4}$. The broken lines having equations $x-vt = \pm\,(v^2/c_2^2-1)^{\frac{1}{2}}\,y$, are lines across which the stress is discontinuous. The numbers attached to the curves give the values of the quantity $\pi a\tau/F$, where $a$ is some characteristic length.

When the velocity $v$ is greater than $c_2$, but less than $c_1$, we see, therefore, that the lines

$$x-vt = \pm\,(\alpha_2^2-1)^{\frac{1}{2}}\,y \tag{13·7}$$

are lines across which the stress is discontinuous. This is due to the presence of the Dirac delta function in the above solution. This naturally leads to discontinuities along the same lines in the pattern of isochromatics (cf. figure 5, which shows the case $v = 0{\cdot}8c_1$, $\lambda = \mu$) which have now changed in shape a great deal from those obtained in § 11.

A similar analysis holds in the case in which the stresses are set up by a point force moving with uniform velocity $u > c_2$ along a line perpendicular to the direction in which the force acts. For a force of magnitude $P$ we have, in the notation of § 10, the following expressions for the components of stress:

$$\sigma_x+\sigma_y=-\frac{(\beta^2-1)\,(1-\beta_1^2)^{\frac{1}{2}}\,xP}{\pi\beta^2\{(1-\beta_1^2)\,x^2+(y-ut)^2\}}, \tag{13·8}$$

$$\sigma_x-\sigma_y=-\frac{2P}{\pi\beta_2^2}\left[\frac{(1-\frac{1}{2}\beta_1^2)\,(1-\beta_1^2)^{\frac{1}{2}}\,x}{(1-\beta_1^2)\,x^2+(y-ut)^2}-\tfrac{1}{2}\pi(y-ut)\,\delta\{(\beta_2^2-1)\,x^2-(y-ut)^2\}\right], \tag{13·9}$$

$$\tau_{xy}=\frac{P}{\pi\beta_2^2}\left[\frac{(1-\beta_1^2)^{\frac{1}{2}}\,(y-ut)}{(1-\beta_1^2)\,x^2+(y-ut)^2}-\tfrac{1}{2}\pi(y-ut)\,\delta\{(\beta_2^2-1)\,x^2-(y-ut)^2\}\right]. \tag{13·10}$$

Again, as is expected from the previous case, we have discontinuities along the lines

$$y-ut=\pm(\beta_2^2-1)^{\frac{1}{2}}\,x. \tag{13·11}$$

These lines are shown in figure 6 which shows the isochromatics in a typical case ($\beta_1=0{\cdot}8$, $\lambda=\mu$).

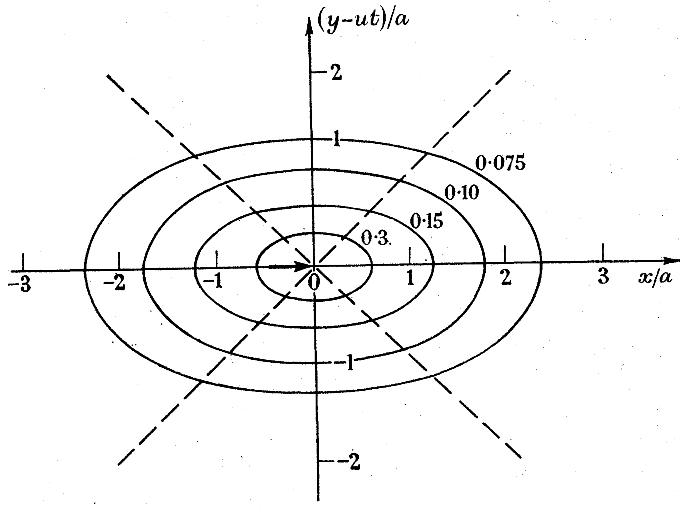


FIGURE 6. The variation of the maximum shearing stress $\boldsymbol{\tau}$ in the case of a point force of magnitude $P$ moving with velocity $v=0{\cdot}8c_1$ at right angles to the direction in which it acts in a medium with Poisson ratio $\frac{1}{4}$. The broken lines, having equations $y-ut=\pm(v^2/c^2-1)^{\frac{1}{2}}\,x$, are lines across which the stress is discontinuous. The numbers attached to the curves give the values of the quantity $\pi a\boldsymbol{\tau}/P$ where $a$ is some characteristic length.

## III. THREE-DIMENSIONAL PROBLEMS

### 14. THE STRESSES PRODUCED BY A POINT FORCE MOVING WITH UNIFORM VELOCITY ALONG THE LINE IN WHICH IT ACTS

In this section and the next we shall consider the solution of particular three-dimensional problems using rectangular Cartesian co-ordinates. We shall make use of the general solution derived in § 3. In both the applications we shall suppose that the body force is of the form $\mathbf{F}=(0,0,Z)$. We shall also write $(u,v,w)$ for the components of the displacement and take $x_1=x$, $x_2=y$, $x_3=z$. The components of the stress tensor will be written, in the usual way, $\tau^{11}=\sigma_x$, $\tau^{22}=\sigma_y$, $\tau^{33}=\sigma_z$, $\tau^{12}=\tau^{21}=\tau_{xy}$, $\tau^{13}=\tau^{31}=\tau_{xz}$, $\tau^{23}=\tau^{32}=\tau_{yz}$.

With this notation, equations (3·7) assume the forms

$$u = -\frac{\beta^2-1}{4\pi^2 c_1^2}\int_{W_4}\frac{\xi\zeta\bar{Z}\,e^{-i\Omega}\,dW}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}, \tag{14·1}$$

$$v = -\frac{\beta^2-1}{4\pi^2 c_1^2}\int_{W_4}\frac{\eta\zeta\bar{Z}\,e^{-i\Omega}\,dW}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}, \tag{14·2}$$

$$w = \frac{1}{4\pi^2 c_1^2}\int_{W_4}\frac{\{\beta^2(\gamma^2-\omega^2)-(\beta^2-1)\zeta^2\}\bar{Z}\,e^{-i\Omega}\,dW}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}, \tag{14·3}$$

where $dW = d\xi\, d\eta\, d\zeta\, d\omega$, $\gamma^2 = \xi^2+\eta^2+\zeta^2$, $W_4$ is the entire $\xi\eta\zeta\omega$-space and $\Omega$ denotes the inner product

$$\Omega = \xi x + \eta y + \zeta z + \omega\tau. \tag{14·4}$$

Similarly from equations (3·14) we derive the relations

$$\sigma_x+\sigma_y+\sigma_z = -\frac{(3\beta^2-4)\rho}{4\pi^2\beta^2}\int_{W_4}\frac{i\zeta\bar{Z}\,e^{-i\Omega}\,dW}{\gamma^2-\omega^2}, \tag{14·5}$$

$$\sigma_x-\sigma_y = \frac{(\beta^2-1)\rho}{2\pi^2\beta^2}\int_{W_4}\frac{i(\xi^2-\eta^2)\zeta\bar{Z}\,e^{-i\Omega}\,dW}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}, \tag{14·6}$$

$$\sigma_x-\sigma_z = \frac{\rho}{2\pi^2\beta^2}\int_{W_4}\frac{i\zeta\{\beta^2(\gamma^2-\omega^2)+(\beta^2-1)(\xi^2-\zeta^2)\}\bar{Z}\,e^{-i\Omega}\,dW}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}, \tag{14·7}$$

$$\tau_{yz} = -\frac{\rho}{4\pi^2\beta^2}\int_{W_4}\frac{i\eta\{\beta^2(\gamma^2-\omega^2)-2(\beta^2-1)\zeta^2\}\bar{Z}\,e^{-i\Omega}\,dW}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}, \tag{14·8}$$

$$\tau_{xz} = -\frac{\rho}{4\pi^2\beta^2}\int_{W_4}\frac{i\xi\{\beta^2(\gamma^2-\omega^2)-2(\beta^2-1)\zeta^2\}\bar{Z}\,e^{-i\Omega}\,dW}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}, \tag{14·9}$$

$$\tau_{xy} = \frac{\rho(\beta^2-1)}{2\pi^2\beta^2}\int_{W_4}\frac{i\xi\eta\zeta\bar{Z}\,e^{-i\Omega}\,dW}{(\gamma^2-\omega^2)(\gamma^2-\beta^2\omega^2)}, \tag{14·10}$$

by means of which we can derive the components of the stress tensor.

If the body force $Z$ is a point force of magnitude $F$ which moves with velocity $v$ along the $z$-axis then

$$Z = \frac{F}{\rho}\,\delta(z-\alpha_1\tau)\,\delta(x)\,\delta(y), \tag{14·11}$$

where, as before, $\alpha_1 = v/c_1$. The Fourier transform of this function is readily shown to be

$$\bar{Z} = \frac{F}{2\pi\rho}\,\delta(\omega+\alpha_1\zeta); \tag{14·12}$$

substituting from equation (14·12) into equations (14·1), (14·2) and (14·3) and performing the $\omega$-integration we find that

$$u = -\frac{F(\beta^2-1)\,i}{8\pi^3\mu\beta^2}\frac{\partial I_1}{\partial x}, \tag{14·13}$$

$$v = -\frac{F(\beta^2-1)\,i}{8\pi^3\mu\beta^2}\frac{\partial I_1}{\partial y}, \tag{14·14}$$

$$w = \frac{F}{8\pi^3\mu\beta^2}\left\{\beta^2 I_2 - i(\beta^2-1)\frac{\partial I_1}{\partial(z-vt)}\right\}, \tag{14·15}$$

where
$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\zeta \exp[-i\{\xi x+\eta y+\zeta(z-vt)\}]\,d\xi\,d\eta\,d\zeta}{\{\xi^2+\eta^2+(1-\alpha_1^2)\zeta^2\}\{\xi^2+\eta^2+(1-\alpha_2^2)\zeta^2\}}, \tag{14·16}$$

$$I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp[-i\{\xi x+\eta y+\zeta(z-vt)\}]\,d\xi\,d\eta\,d\zeta}{\xi^2+\eta^2+(1-\alpha_2^2)\zeta^2}, \tag{14·17}$$

with $\alpha_2 = v/c_2$.

If we let $\xi = \rho\cos\phi$, $\eta = \rho\sin\phi$, $x = r\cos\theta$, $y = r\sin\theta$ and perform the integrations with respect to $\phi$ and $\zeta$ we find that

$$I_1 = \frac{2\pi^2 i}{\alpha_1^2(\beta^2-1)}\int_0^{\infty} J_0(\rho r)\{e^{-\rho\gamma_2}-e^{-\rho\gamma_1}\}\frac{d\rho}{\rho}, \tag{14·18}$$

where we have put
$$\gamma_{1,2} = \frac{z-vt}{\sqrt{(1-\alpha_{1,2}^2)}}. \tag{14·19}$$

This shows that $I_1$ is a function of $r = \sqrt{(x^2+y^2)}$ and $z-vt$ only and that

$$\frac{\partial I_1}{\partial r} = -\frac{2\pi^2 i}{\alpha_1^2(\beta^2-1)}\int_0^{\infty} J_1(\rho r)\{e^{-\rho\gamma_2}-e^{-\rho\gamma_1}\}\,d\rho.$$

Making use of a well-known integral (Watson 1944, p. 514) we find that

$$\frac{\partial I_1}{\partial r} = \frac{2\pi^2 i}{\alpha_1^2(\beta^2-1)\,r}\left\{\frac{\gamma_2}{\sqrt{(r^2+\gamma_2^2)}}-\frac{\gamma_1}{\sqrt{(r^2+\gamma_1^2)}}\right\}. \tag{14·20}$$

Similarly, it is readily shown that

$$\frac{\partial I_1}{\partial(z-vt)} = -\frac{2\pi^2 i}{\alpha_1^2(\beta^2-1)}\left\{\frac{(1-\alpha_2^2)^{-\frac{1}{2}}}{(r^2+\gamma_2^2)^{\frac{1}{2}}}-\frac{(1-\alpha_1^2)^{-\frac{1}{2}}}{(r^2+\gamma_1^2)^{\frac{1}{2}}}\right\} \tag{14·21}$$

and that
$$I_2 = \frac{2\pi^2}{(1-\alpha_2^2)^{\frac{1}{2}}(r^2+\gamma_2^2)^{\frac{1}{2}}}. \tag{14·22}$$

Inserting the values (14·20), (14·21) and (14·22) into equations (14·13) to (14·15) we obtain the formula

$$u = \frac{xF}{4\pi\mu\alpha_2^2 r^2}\left\{\frac{\gamma_2}{(r^2+\gamma_2^2)^{\frac{1}{2}}}-\frac{\gamma_1}{(r^2+\gamma_1^2)^{\frac{1}{2}}}\right\}, \tag{14·23}$$

$$v = \frac{yF}{4\pi\mu\alpha_2^2 r^2}\left\{\frac{\gamma_2}{(r^2+\gamma_2^2)^{\frac{1}{2}}}-\frac{\gamma_1}{(r^2+\gamma_1^2)^{\frac{1}{2}}}\right\}, \tag{14·24}$$

$$w = -\frac{F}{4\pi\mu\alpha_2^2}\left\{\frac{(1-\alpha_2^2)^{\frac{1}{2}}}{(r^2+\gamma_2^2)^{\frac{1}{2}}}-\frac{(1-\alpha_1^2)^{-\frac{1}{2}}}{(r^2+\gamma_1^2)^{\frac{1}{2}}}\right\}, \tag{14·25}$$

by means of which the components of the displacement vector may be calculated. The components of stress may similarly be calculated by means of equations (14·5) to (14·10). We shall not repeat these calculations here since it is easier to treat this problem by the methods of § 6, and we shall do this later (§ 17 below).

## 15. The stresses produced by a point force moving with uniform velocity perpendicular to the direction of the force

We shall now consider the case of a point force of magnitude $P$ acting in the $z$-direction, and with a point of application which is moving with uniform velocity $u$ along the $x$-axis. We then have

$$Z = \frac{P}{\rho}\,\delta(x-\beta_1\tau)\,\delta(y)\,\delta(z), \tag{15·1}$$

where $\beta_1 = u/c_1$. The Fourier transform of this body force is readily seen to be

$$\bar{Z} = \frac{P}{2\pi\rho}\,\delta(\omega+\beta_1\xi); \tag{15·2}$$

substituting from equation (15·2) into equations (14·1) to (14·3) we find that the components of the displacement vector are given by the formulae

$$u = \frac{(\beta^2-1)\,P}{8\pi^3\mu\beta^2}\,\frac{\partial^2 I_1}{\partial x\,\partial z}, \tag{15·3}$$

$$v = \frac{(\beta^2-1)\,P}{8\pi^3\mu\beta^2}\,\frac{\partial^2 I_1}{\partial y\,\partial z}, \tag{15·4}$$

$$w = \frac{P}{8\pi^3\mu\beta^2}\left\{\beta^2 I_2+(\beta^2-1)\,\frac{\partial^2 I_1}{\partial z^2}\right\}, \tag{15·5}$$

where have we denoted by $I_1$, $I_2$ the integrals

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left[-\mathrm{i}\{\xi(x-ut)+\eta y+\zeta z\}\right]\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta}{\{(1-\beta_1^2)\,\xi^2+\eta^2+\zeta^2\}\{(1-\beta_2^2)\,\xi^2+\eta^2+\zeta^2\}}, \tag{15·6}$$

$$I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left[-\mathrm{i}\{\xi(x-ut)+\eta y+\zeta z\}\right]\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta}{(1-\beta_2^2)\,\xi^2+\eta^2+\zeta^2}. \tag{15·7}$$

If, in the integral $I_1$ we let $\eta = \rho\cos\phi$, $\zeta = \rho\sin\phi$, $y = r\cos\theta$, $z = r\sin\theta$ and perform the integrations with respect to $\phi$, $\xi$ we find that $I_1$ is a function of $x-ut$ and $r$ only and that

$$\frac{\partial I_1}{\partial r} = \frac{2\pi^2}{\beta_1^2(\beta^2-1)}\int_0^{\infty}\{(1-\beta_2^2)^{\frac{1}{2}}\,\mathrm{e}^{-\rho\delta_2}-(1-\beta_1^2)^{\frac{1}{2}}\,\mathrm{e}^{-\rho\delta_1}\}\,\frac{J_1(\rho r)}{\rho}\,\mathrm{d}\rho, \tag{15·8}$$

where

$$\delta_{1,2} = \frac{x-ut}{\sqrt{(1-\beta_{1,2}^2)}} \tag{15·9}$$

($\beta_2 = u/c_2$). The integral on the right-hand side of (15·8) is elementary (Watson 1944, p. 514) and we obtain

$$\frac{\partial I_1}{\partial r} = \frac{2\pi^2}{\beta_1^2(\beta^2-1)\,r}\{(1-\beta_2^2)^{\frac{1}{2}}\,(r^2+\delta_2^2)^{\frac{1}{2}}-(1-\beta_1^2)^{\frac{1}{2}}\,(r^2+\delta_1^2)^{\frac{1}{2}}\}, \tag{15·10}$$

where, it will be noted, $r^2 = y^2+z^2$. The integral $I_2$ may be evaluated by a similar procedure:

$$I_2 = \frac{2\pi^2}{(1-\beta_1^2)^{\frac{1}{2}}\,(r^2+\delta_2^2)^{\frac{1}{2}}}, \tag{15·11}$$

substituting from equations (15·10) and (15·11) into equations (15·3), (15·4) and (15·5) we find that

$$u = \frac{(x-ut)\,yP}{4\pi\mu\beta_2^2 r^2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right), \tag{15·12}$$

$$v = -\frac{yzP}{4\pi\mu\beta_2^2 r^4}\left\{R_2-R_1+(x-ut)^2\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\right\}, \tag{15·13}$$

$$w = \frac{P}{4\pi\mu\beta_2^2}\left\{\frac{\beta_2^2}{R_2}+\frac{y^2}{r^4}(R_2-R_1)-\frac{(x-ut)^2 z^2}{r^4}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\right\}, \tag{15·14}$$

where $$R_{1,2}^2 = (1-\beta_{1,2}^2)(y^2+z^2)+(x-ut)^2, \quad r^2 = y^2+z^2. \tag{15·15}$$

In a similar way, if we insert the value (15·2) for $\bar{Z}$ in equations (14·5) to (14·10) and carry out the integrations we find that

$$\sigma_x+\sigma_y+\sigma_z = -\frac{(3\beta^2-4)\,P(1-\beta_1^2)\,z}{4\pi\beta^2 R_1^3}, \tag{15·16}$$

$$\sigma_x-\sigma_y = \frac{Pz}{2\pi\beta_2^2 r^2}\left\{\frac{(R_2-R_1)}{r^2}\left(1-\frac{3y^2}{r^2}\right)+\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\left[1+\frac{(x-ut)^2}{r^2}-\frac{6(x-ut)^2y^2}{r^4}\right]\right.$$
$$\left.-(x-ut)^2\left(\frac{1}{R_2^3}-\frac{1}{R_1^3}\right)\left[1-\frac{(x-ut)^2y^2}{r^4}\right]\right\}, \tag{15·17}$$

$$\sigma_x-\sigma_z = \frac{Pz}{2\pi\beta_2^2 r^2}\left\{\frac{\beta_2^2}{R_2}-\frac{\beta_2^2(x-ut)^2}{R_2^3}+\frac{3y^2}{r^4}(R_2-R_1)-(x-ut)^2\left(\frac{1}{R_2^3}-\frac{1}{R_1^3}\right)\left[1-\frac{(x-ut)^2z^2}{r^4}\right]\right.$$
$$\left.+\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\left[1+\frac{2(x-ut)^2}{r^2}-\frac{5z^2(x-ut)^2}{r^4}+\frac{y^2(x-ut)^2}{r^4}\right]\right\}, \tag{15·18}$$

$$\tau_{yz} = \frac{Py}{4\pi\beta_2^2 r^2}\left\{\frac{\beta_2^2(x-ut)^2}{R_2^3}-\frac{\beta_2^2}{R_2}+\frac{R_2-R_1}{r^2}\left(1+\frac{3(z^2-y^2)}{r^2}\right)\right.$$
$$\left.-\frac{(x-ut)^2}{r^2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\left(1-\frac{11z^2-y^2}{r^2}\right)-\frac{2z^2(x-ut)^4}{r^4}\left(\frac{1}{R_2^3}-\frac{1}{R_1^3}\right)\right\}, \tag{15·19}$$

$$\tau_{xz} = \frac{P(x-ut)}{4\pi\beta_2^2 r^2}\left\{-\frac{\beta_2^2 r^2}{R_2^3}+\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\left(1+\frac{y^2-5z^2}{r^2}\right)+\frac{2z^2(x-ut)^2}{r^2}\left(\frac{1}{R_2^3}-\frac{1}{R_1^3}\right)\right\}, \tag{15·20}$$

$$\tau_{xy} = -\frac{P(x-ut)\,yz}{2\pi\beta_2^2 r^4}\left\{3\left(\frac{1}{R_2}-\frac{1}{R_1}\right)-2(x-ut)^2\left(\frac{1}{R_2^3}-\frac{1}{R_1^3}\right)\right\}. \tag{15·21}$$

A number of calculations based on the formulae (15·16) to (15·21) have been performed (Eason 1954) for the values $\beta_1^2 = 0·01$ and $\beta_1^2 = 0·30$ in an elastic solid for which the Lamé constants $\lambda$ and $\mu$ are equal. The first of these cases ($\beta_1 = 0·1$) corresponds to a force which is moving with a velocity $u$ equal to one-tenth of the larger wave velocity $c_1$. In this case it turns out that the numerical values of the components of stress in terms of the parameters $x-ut$, $y$, $z$ are almost identical with the same quantities calculated for a statical force in terms of parameters $x$, $y$, $z$. The variation of one such component of stress is shown in figure 7. In this diagram the values of the stress component $\sigma_x$ in planes $z = constant$ are depicted by plotting the lines in those planes along which $\sigma_x$ is a constant. The curves shown in figure 7 give the values of $4\pi\sigma_x a^2/F$ in the plane $z = a$, but it is easily seen how similar contours in planes parallel to this one can be calculated directly from this diagram.

In figure 8 we have depicted these contours in the case of a much higher velocity—that corresponding to $\beta_1^2 = 0{\cdot}3$. As we should expect, the values of the component $\sigma_x$ differ greatly from those obtained in the statical case.

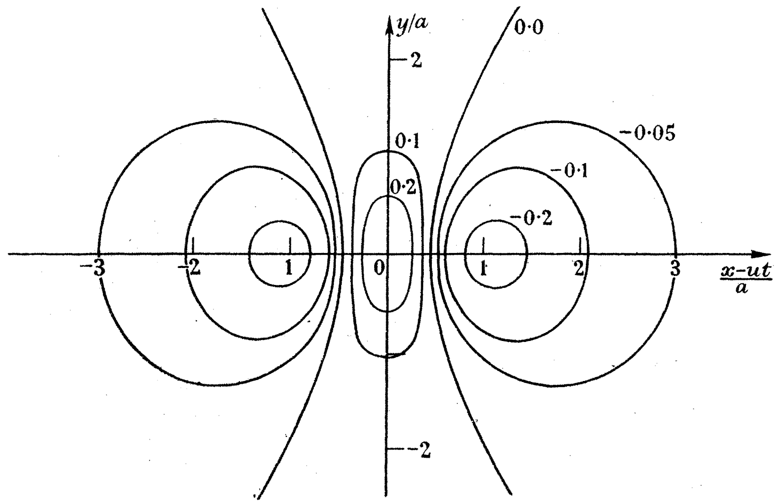


FIGURE 7. The variation of $\sigma_x$ with $x-ut$ and $y$ in the plane $z = a$, where $a$ is some characteristic length, in the case of a point force of magnitude $P$ which acts in the $z$-direction and whose point of application is moving along the $x$-axis with uniform velocity $u = 0{\cdot}1c_1$. The numbers attached to the curves give the values of the dimensionless quantity $4\pi a^2\sigma_x/P$. The value of $\sigma_x$ in any plane parallel to $z = a$ is easily obtained by using the fact that if $x-ut$ and $y$ are fixed $\sigma_x$ is inversely proportional to $z^2$. The Poisson ratio of the material is taken to be $\frac{1}{4}$.

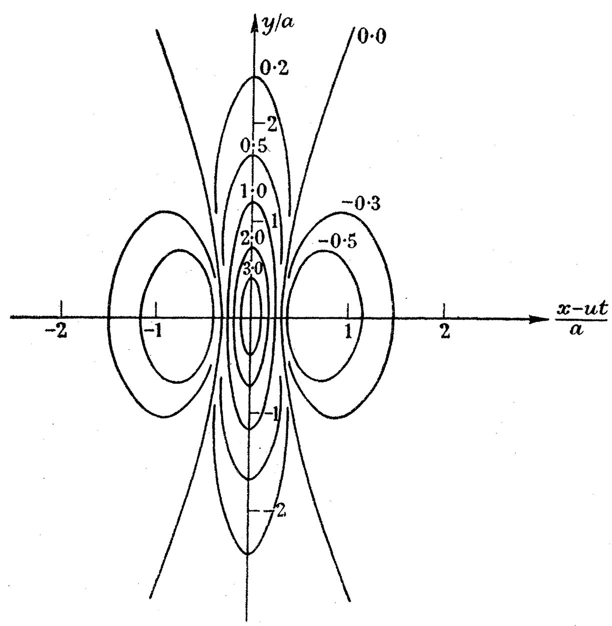


FIGURE 8. The variation of $\sigma_x$ with $x-ut$ and $y$ in the plane $z = a$, where $a$ is some characteristic length, in the case of a point force of magnitude $P$ which acts in the $z$-direction and whose point of application is moving along the $x$-axis with uniform velocity $u = \beta_1 c_1$, where $\beta_1^2 = 0{\cdot}3$. The values of the dimensionless quantity $4\pi a^2\sigma_x/P$ are attached to the appropriate curves. The value of $\sigma_x$ in any plane parallel to $z = a$ is easily obtained by using the fact that if $x-ut$ and $y$ are fixed $\sigma_x$ is inversely proportional to $z^2$. It is assumed that the solid has a Poisson ratio equal to $\frac{1}{4}$.

The reader who is interested in the numerical values of the other components of the stress tensor and in the variation of stress throughout the solid is referred to Eason's thesis (Eason 1954) where these matters are treated in great detail.

## 16. The effect of a circular disk of pressure moving with uniform velocity at right angles to the direction in which it acts

We shall consider the case of a disk of pressure of magnitude $P$, the radius of the disk being $a$, which acts in the $z$-direction but whose centre moves with uniform velocity $u$ along the $x$-axis. We then have for the body force $Z$,

$$Z = \begin{cases} \dfrac{P}{\rho}\delta(z) & \text{if} \quad (x-\beta_1\tau)^2+y^2<a^2, \\ 0 & \text{if} \quad (x-\beta_1\tau)^2+y^2>a^2, \end{cases}$$

where $\beta_1 = u/c_1$. The Fourier transform of this function is, by definition,

$$\overline{Z} = \frac{P}{4\pi^2\rho}\int_{-\infty}^{\infty}\delta(z)\,\mathrm{e}^{\mathrm{i}\zeta z}\,\mathrm{d}z\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau\int_{-a+\beta_1\tau}^{a+\beta_1\tau}\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}x\int_{-[a^2-(x-\beta_1\tau)^2]^{\frac{1}{2}}}^{[a^2-(x-\beta_1\tau)^2]^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\eta y}\,\mathrm{d}y.$$

By changing the variables from $x$, $y$, $\tau$ to $r$, $\theta$ where

$$x-\beta_1\tau = r\cos\theta, \quad y = r\sin\theta,$$

we find that

$$\overline{Z} = \frac{P}{4\pi^2\rho}\int_{-\infty}^{\infty}\exp\{\mathrm{i}\tau(\omega+\beta_1\xi)\}\,\mathrm{d}\tau\int_0^a r\,\mathrm{d}r\int_0^{2\pi}\exp\{\mathrm{i}\rho' r\cos(\theta-\phi)\}\,\mathrm{d}\theta,$$

where $\rho' = \surd(\xi^2+\eta^2)$ and $\tan\phi = \eta/\xi$. The integrations are now elementary and give

$$\overline{Z} = \frac{Pa\delta(\omega+\beta_1\xi)}{\rho(\xi^2+\eta^2)^{\frac{1}{2}}}\,J_1\{a\surd(\xi^2+\eta^2)\}. \tag{16·1}$$

If, now, we substitute this value for $\overline{Z}$ into equations (14·1) to (14·3) and perform the integration with respect to $\omega$ we obtain the expressions

$$u = -\frac{Pa}{4\pi^2\mu\beta_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\zeta J_1\{a\surd(\xi^2+\eta^2)\}}{\xi(\xi^2+\eta^2)^{\frac{1}{2}}}\left[\frac{1}{(1-\beta_2^2)\,\xi^2+\eta^2+\zeta^2}-\frac{1}{(1-\beta_1^2)\,\xi^2+\eta^2+\zeta^2}\right]$$
$$\times\exp[-\mathrm{i}\{\xi(x-ut)+\eta y+\zeta z\}]\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{16·2}$$

$$v = -\frac{Pa}{4\pi^2\mu\beta_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta\zeta J_1\{a\surd(\xi^2+\eta^2)\}}{\xi^2(\xi^2+\eta^2)^{\frac{1}{2}}}\left[\frac{1}{(1-\beta_2^2)\,\xi^2+\eta^2+\zeta^2}-\frac{1}{(1-\beta_1^2)\,\xi^2+\eta^2+\zeta^2}\right]$$
$$\times\exp[-\mathrm{i}\{\xi(x-ut)+\eta y+\zeta z\}]\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{16·3}$$

$$w = \frac{Pa}{4\pi^2\mu\beta_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{J_1\{a\surd(\xi^2+\eta^2)\}}{\xi^2(\xi^2+\eta^2)^{\frac{1}{2}}}\left[\frac{\xi^2+\eta^2}{(1-\beta_2^2)\,\xi^2+\eta^2+\zeta^2}-\frac{(1-\beta_1^2)\,\xi^2+\eta^2}{(1-\beta_1^2)\,\xi^2+\eta^2+\zeta^2}\right]$$
$$\times\exp[-\mathrm{i}\{\xi(x-ut)+\eta y+\zeta z\}]\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{16·4}$$

The evaluation of these integrals would present a formidable problem in the general case. In the case in which the velocity $u$ is very much less than $c_1$, it is possible to expand the integrand in terms of $\beta_1^2$ and $\beta_2^2$; we then find that the integrals, which arise as coefficients in this expansion, may be evaluated in closed form.

For instance, if we expand the integrand in equation (16·2) in powers of $\beta_1^2$ we find

$$u = -\frac{Pa}{4\pi^2\mu\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\xi\zeta J_1\{a\sqrt{(\xi^2+\eta^2)}\}}{(\xi^2+\eta^2)^{\frac{1}{2}}(\xi^2+\eta^2+\zeta^2)^2}\left\{(\beta^2-1)+\beta_1^2\frac{(\beta^4-1)\,\xi^2}{\xi^2+\eta^2+\zeta^2}\right\}$$
$$\times\exp\left[-\mathrm{i}\{\xi(x-ut)+\eta y+\zeta z\}\right]\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

Performing the integrations with respect to $\zeta$ we find that this expression reduces to the form

$$u = \frac{Paz}{8\pi\mu\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{i}\xi J_1\{a\sqrt{(\xi^2+\eta^2)}\}}{\xi^2+\eta^2}\left\{(\beta^2-1)+\beta_1^2\frac{(\beta^4-1)\,\xi^2}{4(\xi^2+\eta^2)}\left[1+z\sqrt{(\xi^2+\eta^2)}\right]\right\}$$
$$\times\exp\left[-\sqrt{(\xi^2+\eta^2)}\,|z|-\mathrm{i}\{\xi(x-ut)+\eta y\}\right]\mathrm{d}\xi\,\mathrm{d}\eta.$$

If we change to new variables $r$, $\theta$, $\rho$, $\phi$ defined by the equations

$$\xi = \rho\cos\phi,\quad \eta = \rho\sin\phi,\quad x-ut = r\cos\theta,\quad y = r\sin\theta$$

and perform the integrations with respect to $\phi$ we find that

$$u = \frac{Paz(x-ut)}{4\mu\beta^2 r}\left\{(\beta^2-1)\,I(1,1;0)+\beta_1^2\frac{(\beta^4-1)}{4r}\left[3I(1,2;-1)-\frac{(x-ut)^2}{r}I(1,3;0)\right.\right.$$
$$\left.\left.+3zI(1,2;0)-\frac{z(x-ut)^2}{r}I(1,3;1)\right]\right\}, \qquad (16·5)$$

where
$$I(m,n;p) = \int_0^{\infty}J_m(a\rho)\,J_n(r\rho)\,\mathrm{e}^{-z\rho}\rho^p\,\mathrm{d}\rho \qquad (16·6)$$

and it will be noted that, here,
$$r^2 = (x-ut)^2+y^2. \qquad (16·7)$$

In a similar fashion we can show that the components $v$ and $w$, may be written in the forms

$$v = \frac{Payz}{4\mu\beta^2 r}\left\{(\beta^2-1)\,I(1,1;0)+\beta_1^2\frac{(\beta^4-1)}{r^4}\left[I(1,2;-1)-\frac{(x-ut)^2}{r}I(1,3;0)\right.\right.$$
$$\left.\left.+zI(1,2;0)-\frac{z(x-ut)^2}{r}I(1,3;1)\right]\right\}, \qquad (16·8)$$

$$w = \frac{Pa}{4\mu\beta^2}\left\{(\beta^2+1)\,I(1,0;-1)+(\beta^2-1)\,zI(1,0;0)+\frac{\beta_1^2}{4r}\left[(3\beta^4+1)\,I(1,1;-2)\right.\right.$$
$$-\frac{(x-ut)^2}{r}(3\beta^4+1)\,I(1,2;-1)+z(3\beta^4+1)\,I(1,1;-1)$$
$$\left.\left.-(3\beta^4+1)\frac{z(x-ut)^2}{r}I(1,2;0)+z^2I(1,1;0)-\frac{z^2(x-ut)^2}{r}I(1,2;1)\right]\right\}. \qquad (16·9)$$

Integrals of the type (16·6) have been evaluated in closed form and tables of values published recently (Eason, Noble & Sneddon 1955), so that the values of the components of the displacement vector can be calculated at every point of the same solid. The components of the stress tensor can be calculated in the same way.

Putting $\beta_1 = 0$ in equations (16·5), (16·8) and (16·9) we obtain

$$u = \frac{Pa(\beta^2-1)\,xz}{4\mu\beta^2 r}I(1,1;0),\quad v = \frac{Pa(\beta^2-1)\,yz}{4\mu\beta^2 r}I(1,1;0),\quad w = \frac{Pa(\beta^2+1)}{4\mu\beta^2}I(1,0;-1) \qquad (16·10)$$

as the solution of the corresponding statical problem. In the equations (16·10) $r = \sqrt{(x^2+y^2)}$.

### 17. The effect of a circular disk of pressure moving with uniform velocity in the direction in which it acts

We shall now consider the case of a uniform pressure $P$ exerted normally over a disk of radius $a$ which is moving with uniform velocity $v$ in the direction of the $z$-axis. The solution of this problem could be obtained by the method of § 14 but, since there is axial symmetry in this case, it is more suitable to make use of the general solution obtained in § 6. In the notation of that section,

$$Z = \begin{cases} \dfrac{P}{\rho}\delta(z-\beta_1\tau) & (0 \leqslant r \leqslant a), \\ 0 & (r > a), \end{cases}$$

where $\beta_1 = v/c_1$, and so

$$\bar{Z} = \frac{Pa}{\rho\xi} J_1(a\xi)\,\delta(\omega+\beta_1\zeta). \tag{17·1}$$

When this expression for $\bar{Z}$ is substituted into equation (6·13) and the elementary integration with respect to $\omega$ performed we find that the component of the displacement vector in the $r$-direction is

$$u = \frac{Pa(\beta^2-1)}{2\pi\mu\beta^2}\int_0^\infty \xi J_1(\xi a)\,J_1(\xi r)\,\mathrm{d}\xi \int_{-\infty}^{\infty} \frac{\mathrm{i}\zeta\exp\{-\mathrm{i}\zeta(z-vt)\}\,\mathrm{d}\zeta}{\{\xi^2+(1-\beta_1^2)\,\zeta^2\}\{\xi^2+(1-\beta_2^2)\,\zeta^2\}}. \tag{17·2}$$

The integration with respect to $\zeta$ may be performed as a result of a contour integration to yield the result

$$u = -\frac{Pa}{2\mu\beta_2^2}\int_0^\infty \frac{J_1(\xi a)\,J_1(\xi r)}{\xi}\{\mathrm{e}^{-\gamma_2\xi}-\mathrm{e}^{-\gamma_1\xi}\}\,\mathrm{d}\xi, \tag{17·3}$$

where

$$\gamma_{1,2}^2 = \frac{(z-vt)^2}{1-\beta_{1,2}^2}. \tag{17·4}$$

By a similar procedure we can show that the $z$-component of displacement is given by

$$w = -\frac{Pa}{2\mu\beta_2^2}\int_0^\infty \frac{J_1(\xi a)\,J_0(\xi r)}{\xi}\{(1-\beta_2^2)^{\frac{1}{2}}\,\mathrm{e}^{-\gamma_2\xi}-(1-\beta_1^2)^{-\frac{1}{2}}\,\mathrm{e}^{-\gamma_1\xi}\}\,\mathrm{d}\xi \tag{17·5}$$

and that the components of the stress tensor are determined by the equations

$$\sigma_r+\sigma_\theta+\sigma_z = -\frac{(3\beta^4-4)\,Pa}{2\beta^2(1-\beta_1^2)}\int_0^\infty J_1(\xi a)\,J_0(\xi r)\,\mathrm{e}^{-\gamma_1\xi}\,\mathrm{d}\xi, \tag{17·6}$$

$$\sigma_z = \frac{Pa}{\beta_2^2}\int_0^\infty J_1(\xi a)\,J_0(\xi r)\left\{\mathrm{e}^{-\gamma_2\xi}-\frac{(1-\beta_1^2+\frac{1}{2}\beta_2^2)}{1-\beta_1^2}\,\mathrm{e}^{-\gamma_1\xi}\right\}\mathrm{d}\xi, \tag{17·7}$$

$$\sigma_r = -\frac{Pa}{\beta_2^2}\int_0^\infty J_1(\xi a)\,J_0(\xi r)\left\{\mathrm{e}^{-\gamma_2\xi}-\frac{1-\frac{1}{2}\beta_2^2}{1-\beta_1^2}\mathrm{e}^{-\gamma_1\xi}\right\}\mathrm{d}\xi + \frac{Pa}{\beta_2^2}\int_0^\infty \frac{J_1(\xi a)\,J_1(\xi r)}{r\xi}\{\mathrm{e}^{-\gamma_2\xi}-\mathrm{e}^{-\gamma_1\xi}\}\,\mathrm{d}\xi, \tag{17·8}$$

$$\tau_{rz} = \frac{Pa}{\beta_2^2}\int_0^\infty J_1(\xi a)\,J_1(\xi r)\left\{\frac{1-\frac{1}{2}\beta_2^2}{(1-\beta_2^2)^{\frac{1}{2}}}\,\mathrm{e}^{-\gamma_2\xi}-(1-\beta_1^2)^{-\frac{1}{2}}\,\mathrm{e}^{-\gamma_1\xi}\right\}\mathrm{d}\xi. \tag{17·9}$$

The integrals occurring on the right-hand side of these equations are all integrals of the type (16·6) with $\gamma_1$, $\gamma_2$ replacing $z$. They may therefore be evaluated by the methods of the paper cited above (Eason *et al.* 1955). If the tables provided in that paper are to be used for the calculation of the components of stress (or displacement) at a point then it will usually be found necessary to interpolate between the tabulated values.

The tables may be used directly in the case of very small velocities $v$. If $v \ll c_1$ then it is readily shown that, if we neglect terms of order $\beta_1^4$, the components of stress may be calculated from the equations

$$\sigma_r+\sigma_\theta+\sigma_z = -\frac{(3\beta^2-4)\,Pa}{2\beta^2}\{I(1,0;\,0)+\beta_1^2[I(1,0;\,0)-\tfrac{1}{2}zI(1,0;\,1)]\}, \tag{17·10}$$

$$\begin{aligned}\sigma_z = -\frac{Pa}{2\beta^2}\{&\beta^2 I(1,0;\,0)+(\beta^2-1)\,zI(1,0;\,1)\\&+\beta_1^2[\beta^2 I(1,0;\,0)+\tfrac{1}{4}(3\beta^4-2\beta^2-3)\,zI(1,0;\,1)-\tfrac{1}{4}(\beta^4-1)\,z^2I(1,0;\,2)]\},\end{aligned} \tag{17·11}$$

$$\begin{aligned}\sigma_r = -\frac{Pa}{2\beta^2}\Big\{&(\beta^2-2)\,I(1,0;\,0)-(\beta^2-1)\,zI(1,0;\,1)+(\beta^2-1)\frac{z}{r}I(1,1;\,0)\\&+\beta_1^2\Big[(\beta^2-2)\,I(1,0;\,0)-\tfrac{1}{4}(3\beta^4+2\beta^2-7)\,zI(1,0;\,1)+(\beta^4-1)\,z^2I(1,0;\,2)\\&+\tfrac{3}{4}(\beta^4-1)\frac{z}{r}I(1,1;\,0)-\frac{z^2}{4r}(\beta^4-1)\,I(1,1;\,1)\Big]\Big\},\end{aligned} \tag{17·12}$$

$$\begin{aligned}\tau_{rz} = -\frac{Pa}{2\beta^2}\{&I(1,1;\,0)+(\beta^2-1)\,zI(1,1;\,1)-\beta_1^2[\tfrac{1}{4}(\beta^4-3)\,I(1,1;\,0)\\&-\tfrac{1}{4}(3\beta^4-5)\,zI(1,1;\,1)+\tfrac{1}{4}(\beta^4-1)\,z^2I(1,1;\,2)]\},\end{aligned} \tag{17·13}$$

where the integral $I(m,n;\,p)$ is defined by equation (16·6).

The solution of the problem of the point force of magnitude $F$ moving uniformly in the direction in which it acts may be deduced from equations (17·3) to (17·9) by putting $P = F/\pi a^2$ in these expressions and letting $a \to 0$. If we make use of the fact that

$$\lim_{a\to 0}\frac{J_1(\xi a)}{a} = \tfrac{1}{2}\xi,$$

we obtain the equations

$$\sigma_r+\sigma_\theta+\sigma_z = -\frac{(3\beta^2-4)\,F}{4\pi\beta^2(1-\beta_1^2)}\int_0^\infty \xi J_0(\xi r)\,e^{-\gamma_1\xi}\,d\xi,$$

$$\sigma_z = \frac{F}{2\pi\beta_2^2}\int_0^\infty \xi J_0(\xi r)\left\{e^{-\gamma_2\xi}-\frac{(1-\beta_1^2+\frac{1}{2}\beta_2^2)}{1-\beta_1^2}\,e^{-\gamma_1\xi}\right\}d\xi,$$

$$\sigma_r = -\frac{F}{2\pi\beta_2^2}\int_0^\infty \xi J_0(\xi r)\left\{e^{-\gamma_2\xi}-\frac{(1-\frac{1}{2}\beta_2^2)}{1-\beta_1^2}\,e^{-\gamma_1\xi}\right\}d\xi + \frac{F}{2\pi\beta_2^2 r}\int_0^\infty J_1(\xi r)\,\{e^{-\gamma_2\xi}-e^{-\gamma_1\xi}\}\,d\xi,$$

$$\tau_{rz} = \frac{F}{2\pi\beta_2^2}\int_0^\infty \xi J_1(\xi r)\left\{\frac{1-\frac{1}{2}\beta_2^2}{(1-\beta_2^2)^{\frac{1}{2}}}\,e^{-\gamma_2\xi}-(1-\beta_1^2)^{-\frac{1}{2}}\,e^{-\gamma_1\xi}\right\}d\xi$$

for the determination of the components of the stress tensor.

The values of these integrals are known (Watson 1944, p. 416) and give the results:

$$\sigma_r+\sigma_\theta+\sigma_z = -\frac{(3\beta^2-4)(z-vt)F}{4\pi\beta^2R_1^3}, \tag{17·14}$$

$$\sigma_z = \frac{(z-vt)F}{2\pi\beta_2^2}\left\{\frac{1-\beta_2^2}{R_2^3}-\frac{1-\beta_1^2+\frac{1}{2}\beta_2^2}{R_1^3}\right\}, \tag{17·15}$$

$$\sigma_r = -\frac{(z-vt)F}{2\pi\beta_2^2}\left\{\frac{1-\beta_2^2}{R_2^3}+\frac{1}{r^2R_2}-\frac{1-\frac{1}{2}\beta_2^2}{R_1^3}-\frac{1}{r^2R_1}\right\}, \tag{17·16}$$

$$\tau_{rz} = \frac{rF}{2\pi\beta_2^2}\left\{\frac{(1-\beta_2^2)(1-\frac{1}{2}\beta_2^2)}{R_2^3}-\frac{1-\beta_1^2}{R_1^3}\right\}, \tag{17·17}$$

where we have put

$$R_{1,2}^2 = (z-vt)^2+(1-\beta_{1,2}^2)\,r^2. \tag{17·18}$$

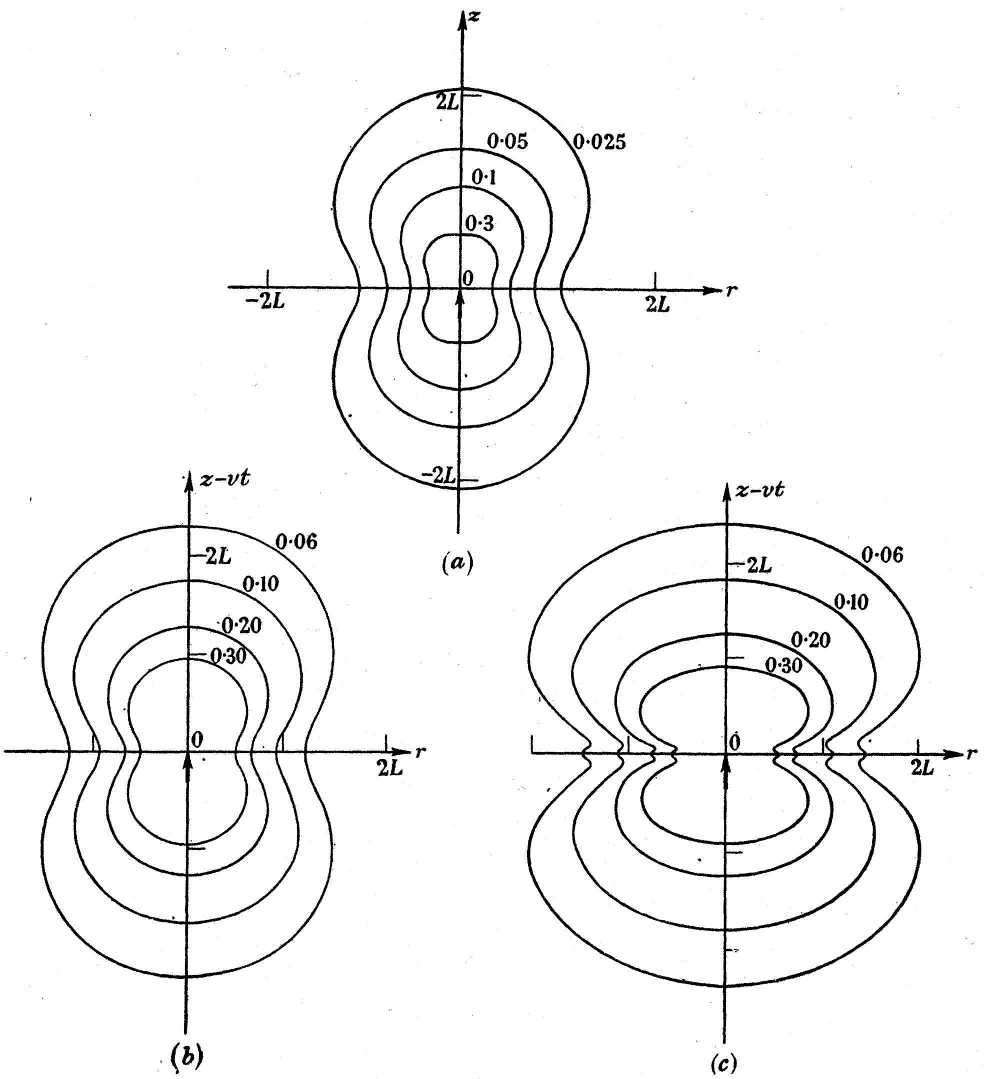


FIGURE 9. The variation of the maximum shearing stress $\tau$ produced in a solid with Poisson ratio $\frac{1}{4}$ by a point force of magnitude $F$ moving in the direction in which it acts with uniform velocity $v$. The numbers attached to the curves give the values of the dimensionless quantity $\pi\tau L^2/F$, where $L$ is a characteristic length. Curves $(a)$ correspond to $v = 0$, $(b)$ to $v = 0{\cdot}10c_1$, and $(c)$ to $v = 0{\cdot}52c_1$.

From these expressions it is a simple matter to calculate the maximum shearing stress $\tau$ at the points with cylindrical co-ordinates $r$, $z$ at any time $t$. The locus of points for which $\tau$ = constant will be a surface of revolution which moves in the direction of the axis of symmetry. The isochromatic lines formed by the intersection of these surfaces with a plane $\theta = constant$ are shown in figure 9 for three values of the velocity $v$: (*a*) statical case, $v = 0$, (*b*) $v = 0{\cdot}10c_1$, (*c*) $v = 0{\cdot}52c_1$. In all three cases it is assumed that the Poisson ratio of the elastic material is 0·25, i.e. that $\lambda = \mu$. In the case $v = 0{\cdot}10c_1$ we see that the isochromatic lines are almost the same as those in the statical case, showing that quite a high velocity must be attained by the moving force before an appreciable change is observed in the stress pattern.

## 18. The stresses produced by a periodic point force

We shall now consider the distribution of stress produced by a point force of maximum magnitude $F$, pointing in the $z$-direction, and varying harmonically with the time. If we take cylindrical co-ordinates $r$, $z$ with origin at the point of application of the force then, in the notation of § 6, we have

$$Z = \frac{F}{\rho} e^{iq\tau} \frac{\delta(z)\,\delta(r)}{2\pi r} \quad (q = p/c_1), \tag{18·1}$$

so that

$$\bar{Z} = \frac{F}{2\pi\rho} \delta(\omega + q). \tag{18·2}$$

If we substitute from equation (18·2) into equations (6·13) and (6·14) and perform the integration with respect to $\omega$, which is immediate, we find that the components of the displacement vector may be written in the form

$$u = \frac{F e^{iq\tau}}{4\pi^2\mu\beta^2 q^2} \frac{\partial^2}{\partial r\,\partial z}\{I(\beta q) - I(q)\}, \tag{18·3}$$

$$w = \frac{F e^{iq\tau}}{4\pi^2\mu\beta^2 q^2}\left\{\beta^2 q^2 I(\beta q) + \frac{\partial^2}{\partial z^2}[I(\beta q) - I(q)]\right\}, \tag{18·4}$$

where we have adopted the notation

$$I(q) = \int_0^\infty \xi J_0(\xi r)\, d\xi \int_{-\infty}^{\infty} \frac{e^{-i\zeta z}\, d\zeta}{\xi^2 + \zeta^2 - q^2}.$$

The $\zeta$-integration in $I(q)$ may be effected by means of the calculus of residues and the resulting $\xi$-integration by means of a well-known result in the theory of Bessel functions (Watson 1944, p. 416). We find finally that

$$I(q) = \frac{\pi e^{-iq(r^2+z^2)^{\frac{1}{2}}}}{\sqrt{(r^2+z^2)}}. \tag{18·5}$$

When we insert the value (18·5) for $I(q)$ into equations (18·3) and (18·4) we find that the components of the displacement vector for this problem take the forms

$$u = \frac{rz\, e^{iq\tau} F}{4\pi\mu\beta^2 q^2}\left\{\frac{3}{R^5}(e^{-i\beta qR} - e^{-iqR}) + \frac{3iq}{R^4}(\beta\, e^{-i\beta qR} - e^{-iqR}) - \frac{q^2}{R^3}(\beta^2 e^{-i\beta qR} - e^{-iqR})\right\}, \tag{18·6}$$

$$w = \frac{e^{iq\tau} F}{4\pi\mu\beta^2 q^2}\left\{\frac{\beta^2 q^2}{R} e^{-i\beta qR} + \frac{(2z^2 - r^2)}{R^5}(e^{-i\beta qR} - e^{-iqR}) + \frac{(2z^2 - r^2)\, iq}{R^4}(\beta\, e^{-i\beta qR} - e^{-iqR}) - \frac{zq^2}{R^3}(\beta^2 e^{-i\beta qR} - e^{-iqR})\right\}, \tag{18·7}$$

where $R^2 = r^2 + z^2$.

The stress tensor may now be calculated by making use of equations (6·1).



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## 19. The effect of an impulsive point force

Finally, we consider the effect of an infinite elastic medium of the application at the origin of co-ordinates of an impulsive force, of total impulse $I$, in the $z$-direction. In the notation of § 6 we have

$$Z = \frac{I}{\rho}\delta(t)\frac{\delta(z)\,\delta(r)}{2\pi r} = \frac{Ic_1}{2\pi\rho r}\delta(r)\,\delta(z)\,\delta(\tau), \tag{19·1}$$

and we can derive the solution in this case from that of § 18 by noting that

$$\frac{F}{\rho}\frac{\delta(z)\,\delta(r)}{2\pi r}\int_{-\infty}^{\infty} e^{iq\tau}\,dq = \frac{2\pi F}{\rho}\delta(\tau)\frac{\delta(z)\,\delta(r)}{2\pi r}.$$

Hence the solution in this case is obtained from equations (18·6) and (18·7) by taking $F = Ic_1/(2\pi)$ and integrating with respect to $q$ from $-\infty$ to $\infty$. We therefore find that

$$u = \frac{Ic_1 rz}{8\pi^2\mu\beta^2}\int_{-\infty}^{\infty}\frac{e^{iq\tau}}{q^2}\left\{\frac{3}{R^5}(e^{-i\beta qR}-e^{-iqR})+\frac{3iq}{R^4}(\beta\,e^{-i\beta qR}-e^{-iqR})-\frac{q^2}{R^3}(\beta^2 e^{-i\beta qR}-e^{-iqR})\right\}dq, \tag{19·2}$$

$$w = \frac{Ic_1}{8\pi^2\mu\beta^2}\int_{-\infty}^{\infty}\frac{e^{iq\tau}}{q^2}\left\{\frac{\beta^2q^2}{R}e^{-i\beta qR}+\frac{(2z^2-r^2)}{R^5}(e^{-i\beta qR}-e^{-iqR})\right.$$
$$\left.+\frac{(2z^2-r^2)\,iq}{R^4}(\beta\,e^{-i\beta qR}-e^{-iqR})-\frac{z^2q^2}{R^3}(\beta^2 e^{-i\beta qR}-e^{-iqR})\right\}dq. \tag{19·3}$$

It is readily seen that these integrals contain discontinuities in the shape of Dirac delta functions at the points $\tau = R$, $\tau = \beta R$. In other words, these integrals represent two circular waves of discontinuity moving out from the origin with the velocities $c_1$, $c_2$ of the $P$- and $S$-waves.

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